STRENGTH OF MATERIALS-I

Class: Civil II Year I Sem

STRESS & STRAIN

Unit 1

1.1 LOAD

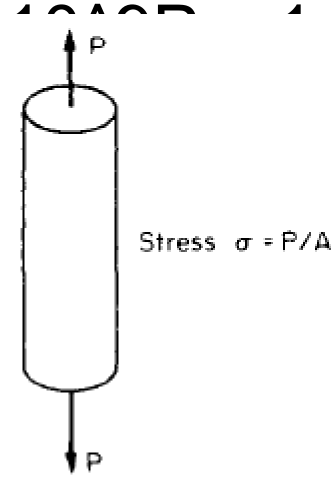
- Load is defined as the set of external forces acting on a mechanism or engineering structure which arise from service conditions in which the components work
- Common loads in engineering applications are tension and compression
- Tension:- Direct pull. Eg:Force present in lifting hoist
- Compression:- Direct push. Eg:- Force acting on the pillar of a building
- Sign convention followed: Tensile forces are positive and compressive negative

1.1.1TYPES OF LOAD

- There are a number of different ways in which load can be applied to a member. Typical loading types are:
- A) **Dead/ Static load**- Non fluctuating forces generally caused by gravity
- B) **Live load-** Load due to dynamic effect. Load exerted by a lorry on a bridge
- C) **Impact load or shock load**- Due to sudden blows
- D) **Fatigue or fluctuating or alternating loads**: Magnitude and sign of the forces changing with time

1.2 STRESS

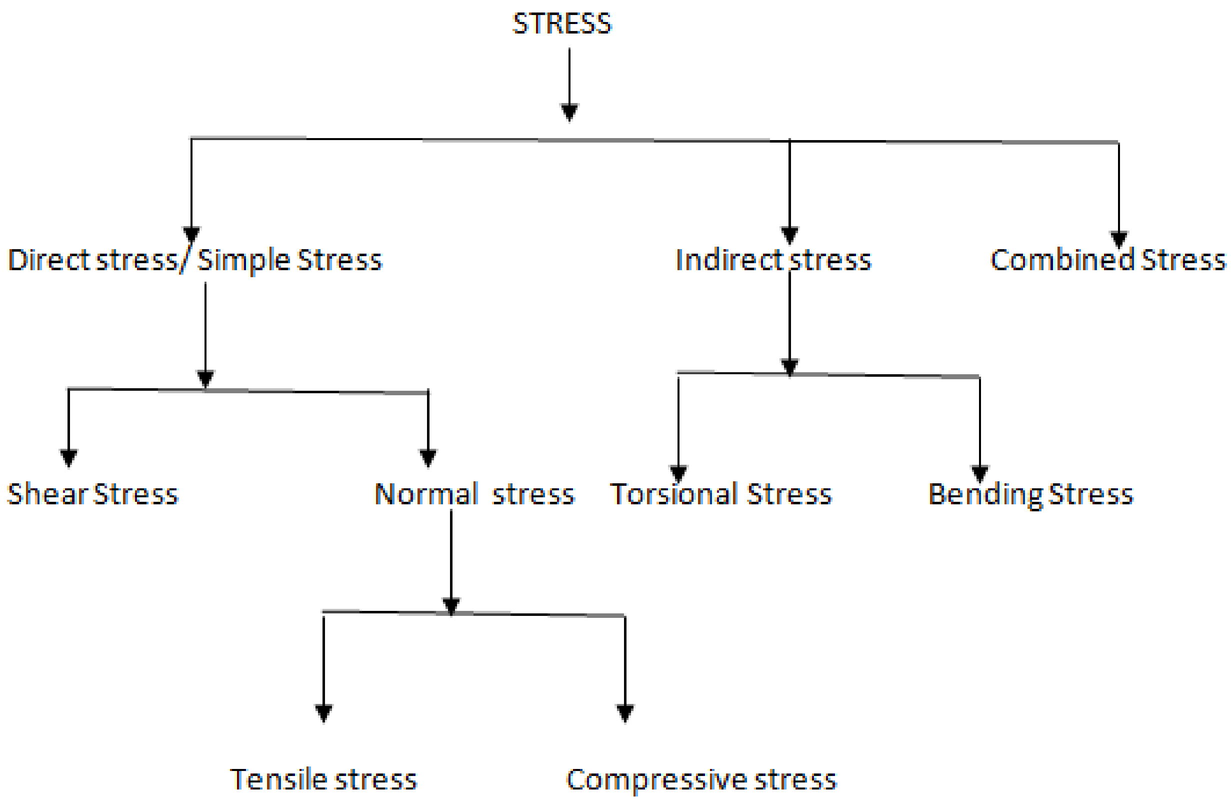
- When a material is subjected to an external force, a resisting force is set up within the component, this internal resistance force per unit area is called stress. SI unit is N/m²(Pa). 1kPa=1000Pa, 1MPa=10^6 Pa, 1 Gpa
Terra Pascal=10^12 Pa
- In engineering applications, we use th
the original cross section area of the s
and it is known as conventional stress
Engineering stress



1.3 STRAIN

- When a body is subjected to some external force, there is some change of dimension of the body. The ratio of change of dimension of the body to its original dimension is known as strain
- Strain is a dimensionless quantity
- Strain may be:- a) Tensile strain b) Compressive strain c) Volumetric strain d) Shear strain
- **Tensile strain-** Ratio of increase in length to original length of the body when it is subjected to a pull force
- **Compressive strain**- Ratio of decrease in length to original length of the body when it is subjected to a push force
- **Volumetric strain-** Ratio of change of volume of the body to the original volume
- **Shear strain-**Strain due to shear stress

1.4 TYPE OF STRESSES



1.4.1TYPES OF DIRECT STRESS

- Direct stress may be normal stress or shear stress
- **Normal stress (σ)** is the stress which acts in direction perpendicular to the area. Normal stress is further classified into tensile stress
- **Tensile stress** is the stress induced in a body, when it is subjected to two equal and opposite pulls (tensile forces) as a result of which there is a tendency in increase in length
- It acts normal to the area and pulls on the area

1.4.1 TYPES OF DIRECT STRESS (Tensile stress)

- Consider a bar subjected to a tensile force P at its ends. Let

  A= Cross sectional area of the body

  L=Original length of the body

  dL= Increase in length of the body due to its pull P

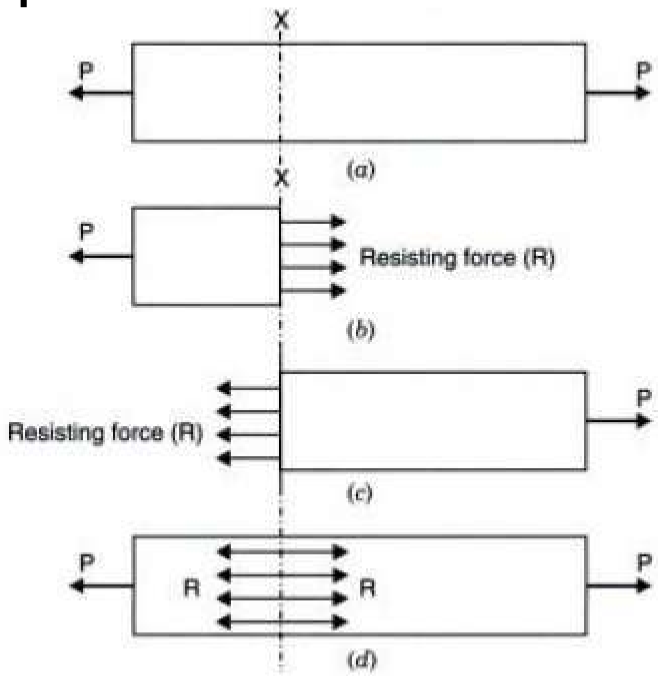
  ς= Stress induced in the body

  e= Tensile strain

Consider a section X-X which divides the body into two halves

1.4.1 TYPES OF DIRECT STRESS (Tensile stress)

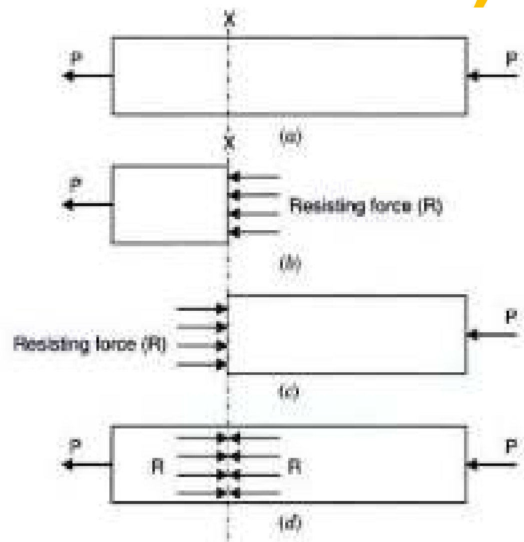
- The left part of the section x-x, will be in equilibrium if P=R (Resisting force). Similarly the right part of the section x-x will be in equilibrium if P=I



1.4.1 TYPES OF DIRECT STRESS (Tensile stress)

- Tensile stress (ς)= Resisting force/ Cross sectional area= Applied force/Cross sectional area=P/A
- Tensile strain= Increase in length/Original length= dL/L
- Compressive stress:- Stress induced in a body, when subjected to two equal and opposite pushes as a result of which there is a tendency of decrease in length of the body
- It acts normal to the area and it pushes on the area
- In some cases the loading situation is such that the stress will vary across any given section. In such cases the stress at any given point is given by
- ς= Lt ΔA→ 0 ΔP/ ΔA= dP/dA= derivative of force w.r.t area

1.4.1 TYPES OF DIRECT STRESS (Compressive stress)



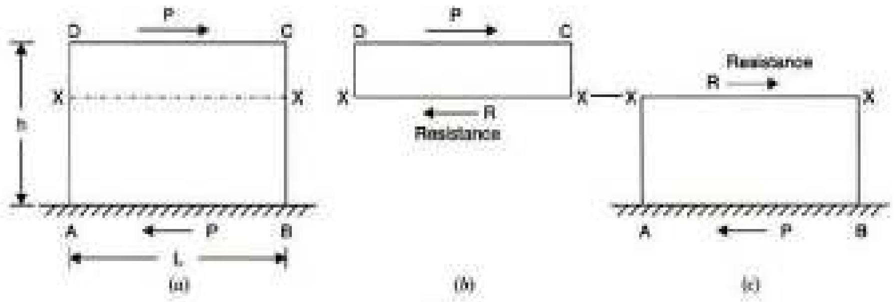
- **Compressive stress**=Resisting force/ cross sectional area= Applied force/ cross sectional area
- Compressive strain= Decrease in length/ Original length= -dL/L
- Sign convention for direct stress and strain:- Tensile stresses and strains are considered positive in sense producing an increase in length. Compressive stresses and strains are considered negative in sense producing decrease in length

1.4.1 TYPES OF DIRECT STRESS (Shear stress)

- **Shear stress** :- Stress Induced in a body, when subjected to two equal and opposite forces which are acting tangentially across the resisting section as a result of which the body tends to shear off across that section
- Consider a rectangular block of height h, length L and width unity. Let the bottom face AB of the block be fixed to the surface as shown. Let P be the tangential force applied along top face CD of the block. For the equilibrium of the block, the surface AB will offer a tangential reaction force R which is equal in magnitude and opposite in direction to the applied tangential force P

1.4.1 TYPES OF DIRECT STRESS (Shear stress)

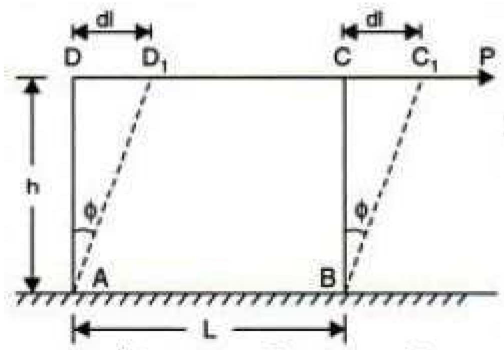
- Consider a section X-X cut parallel to the applied force which splits rectangle into two parts



- For the upper part to be in equilibrium; Applied force P=Resisting force R
- For the lower part to be in equilibrium; Applied force P=Resisting force R
- Hence, shear stress τ= Resisting force/Resisting area=P/L x 1=P/L
- Shear stress is tangential to the area on which it acts

1.4.1 TYPES OF DIRECT STRESS (Shear stress)

- As the face AB is fixed, the rectangular section ABCD will be distorted to ABC1D1, such that new vertical face AD1 makes an angle φ with the initial face AD



(a)

- Angle φ is called shear strain. As φ is very small,
- φ=tan φ=DD1/AD=dl/h
- Hence shear strain=dl/h

1.5 ELASTICITY & ELASTIC LIMIT

- The property of a body by virtue of which it undergoes deformation when subjected to an external force and regains its original configuration (size and shape) upon the removal of the deforming external force is called elasticity.
- The stress corresponding to the limiting value of external force upto and within which the deformation disappears completely upon the removal of external force is called elastic limit
- A material is said to be elastic if it returns to its original, unloaded dimensions when load is removed.
- If the external force is so large that the stress exceeds the elastic limit, the material loses to some extent its property of elasticity. If now the force is removed, the material will not return to its original shape and size and there will be a residual deformation in the material

1.6 HOOKE'S LAW & ELASTIC MODULI

- Hooke's law states that: " When a body is loaded within elastic limit, the stress is proportional to strain developed" or "Within the elastic limit the ratio of stress applied to strain developed is a constant"
- The constant is known as Modulus of elasticity or Elastic modulus or Young's modulus
- Mathematically within elastic limit

Stress/Strain=$ς$/e=E

$ς$= P/A; e

=ΔL/L E=PL/A

ΔL

1.7 HOOKE'S LAW & ELASTIC MODULI

- Young's modulus (E) is generally assumed to be the same in tension or compression and for most of engineering applications has a high numerical value. Typically, E=210 x 10^9 N/m² (=210 GPa) for steel
- Modulus of rigidity, G= τ/φ= Shear stress/ shear strain
- Factor of safety= Ultimate stress/Permissible stress
- In most engineering applications strains donot often exceed 0.003 so that the assumption that deformations are small in relation to orinal dimensions is generally valid

1.8 STRESS-STRAIN CURVE (TENSILE TEST)

- Standard tensile test involves subjecting a circular bar of uniform cross section to a gradually increasing tensile load until the failure occurs
- Tensile test is carried out to compare the strengths of various materials
- Change in length of a selected gauge length of bar is recorded by extensometers
- A graph is plotted with load vs extension or stress vs strain

1.8 STRESS-STRAIN CURVE (TENSILE TEST)

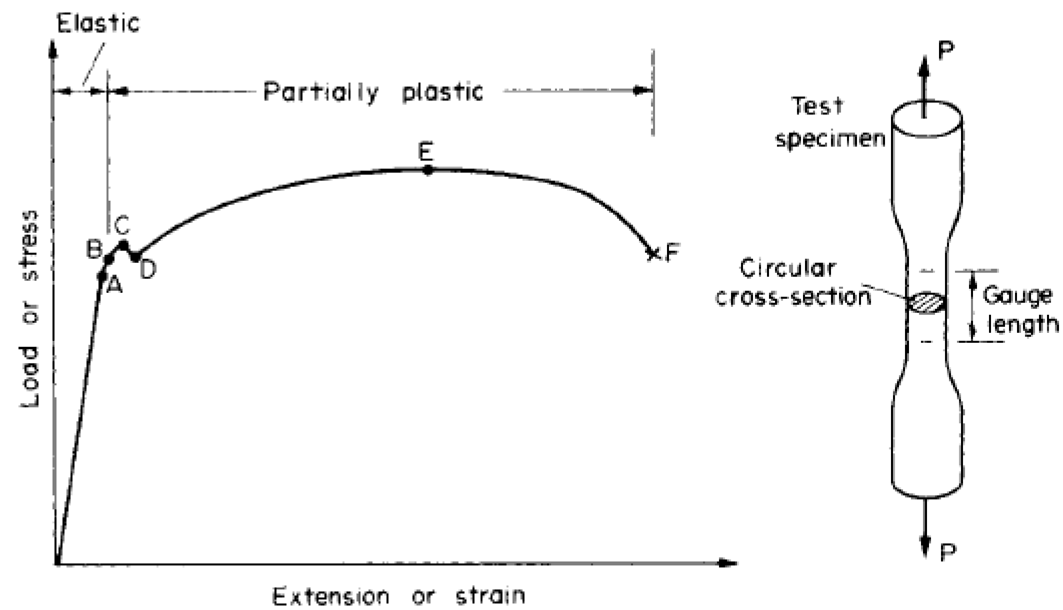


Fig. 1.3. Typical tensile test curve for mild steel.

1.8 STRESS-STRAIN CURVE (TENSILE TEST DIAGRAM)

- A→ Limit of proportionality; It is the point where the linear nature of the stress strain graph ceases
- B→ Elastic limit; It is the limiting point for the condition that material behaves elastically, but hooke's law does not apply . For most practical purposes it can be often assumed that limit of proportionality and elastic limits are the same
- Beyond the elastic limits, there will be some permanent deformation or permanent set when the load is removed
- C (Upper Yield point), D (Lower yield point)→ Points after which strain increases without correspondingly high increase in load or stress
- E→ Ultimate or maximum tensile stress; Point where the necking starts
- F→ Fracture point

RELATIONSHIPS BETWEEN STRESS & STRAIN

- **A) 1-Dimensional case** (due to pull or push or shear force)

  ς=Ee

- **B) 2-Dimensional case**
- Consider a body of length L, width B and height H. Let the body be subjected to an axial load. Due to this axial load, there is a deformation along the length of the body. This strain corresponding to this deformation is called longitudinal strain.
- Similarly there are deformations along directions perpendicular to line of application of fore. The strains corresponding to these deformations are called lateral strains

RELATIONSHIPS BETWEEN STRESS & STRAIN

δL = Increase in length,
δb = Decrease in breadth, and
δd = Decrease in depth.

Then longitudinal strain $= \dfrac{\delta L}{L}$

and lateral strain $= \dfrac{\delta b}{b}$ or $\dfrac{\delta d}{d}$

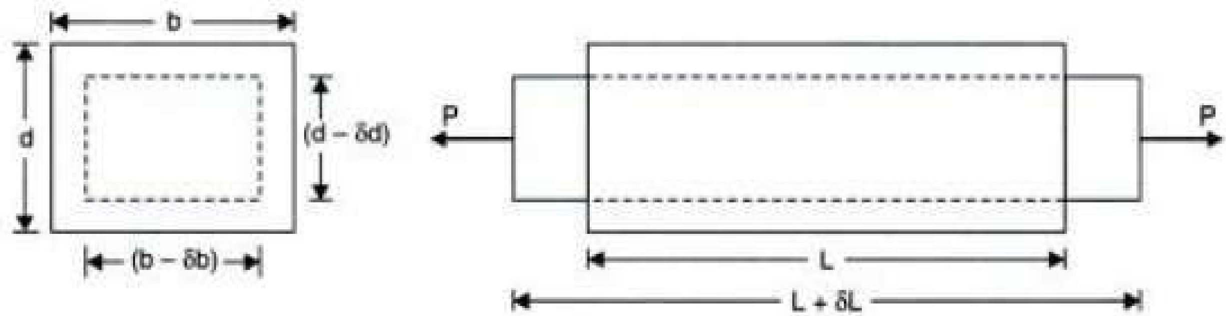


Fig. 1.3. Typical tensile test curve for mild steel.

RELATIONSHIPS BETWEEN STRESS & STRAIN

- Longitudinal strain is always of opposite sign of that of lateral strain. Ie if the longitudinal strain is tensile, lateral strains are compressive and vice versa
- Every longitudinal strain is accompanied by lateral strains in orthogonal directions
- Ratio of lateral strain to longitudinal strain is called Poisson's ratio (μ); Mathematically,
- μ=-Lateral strain/Longitudinal strain
- Consider a rectangular figure ABCD subjected a stress in σx direction and in σ y direction

RELATIONSHIPS BETWEEN STRESS & STRAIN

- Strain along x direction due to $\varsigma x = \varsigma\ x/E$

  Strain along x direction due to $\varsigma\ y = -\mu \times \varsigma y/E$

  Total strain in x direction $ex = \varsigma\ x/E - \mu \times$

  $\varsigma y/E$

  Similarly total strain in y direction, $ey = \varsigma\ y/E - \mu \times \varsigma x/E$
- In the above equation tensile stresses are considered as positive and compressive stresses as negative
- **C) 3 Dimensional case:-**

  Consider a 3 D body subjected to 3 orthogonal normal stresses in x,y and z directions

RELATIONSHIPS BETWEEN STRESS & STRAIN

- Strain along x direction due to ςx= ς x/E
  Strain along x direction due to ς y=-μ x ςy/E Strain along x direction due to ς z=-μ x ςz/E

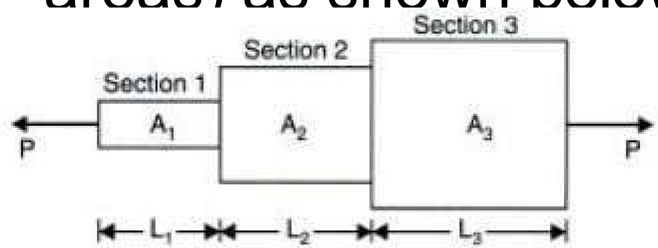
  Total strain in x direction ex= ς x/E - μ x (ςy/E + ςz/E)

  Similarly total strain in y direction, ey= ς y/E - μ x (ςxE + ςz/E)

  Similarly total strain in z direction, ez= ς z/E - μ x (ςxE + ςy/E)

1.10 ANALYSIS OF BARS OF VARYING CROSS SECTION

- Consider a bar of different lengths and of different diameters (and hence of different cross sectional areas) as shown below. Let this bar be subjected to



- The total change in length will be obtained by adding the changes in length of individual sections
- Total stress in section 1: $\varsigma 1 = E1 \times \Delta L1/L1$

  $\varsigma 1 \times L1/E1 = \Delta L1$

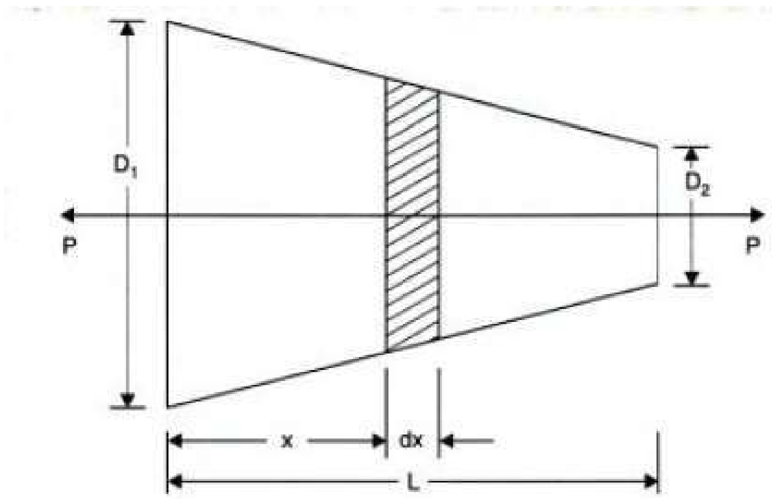
  $\varsigma 1 = P/A1$; Hence $\Delta L1 = PL1/A1E1$
- Similarly, $\Delta L2 = PL2/A2E2$; $\Delta L3 = PL3/A3E3$

1.10 ANALYSIS OF BARS OF VARYING CROSS SECTION

- Hence total elongation $\Delta L = P x (L1/A1E1 + L2/A2E2 + L3/A3E3)$
- If the Young's modulus of different sections are the same, $E1 = E2 = E3 = E$; Hence $\Delta L = P/E x (L1/A1 + L2/A2 + L3/A3)$
- When a number of loads are acting on a body, the resulting strain, according to principle of superposition, will be the algebraic sum of strains caused by individual loads
- While using this principle for an elastic body which is subjected to a number of direct forces (tensile or compressive) at different sections along the length of the body, first the free body diagram of individual section is drawn. Then the deformation of each section is calculated and the total deformation is equal to the algebraic sum of deformations of individual sections

1.11 ANALYSIS OF UNIFORMLY TAPERING CIRCULAR ROD

- Consider a bar uniformly tapering from a diameter D1 at one end to a diameter D2 at the other end
- Let
- P→ Axial load acting on the bar
- L→ Length of bar
- E→ Young's modulus of the material



1. 11 ANALYSIS OF UNIFORMLY TAPERING CIRCULAR ROD

- Consider an infinitesimal element of thickness dx, diameter Dx at a distance x from face with diameter D1.

  Deformation of the element $d(\Delta x) = P \times dx / (Ax \times E)$

  $Ax = \pi/4 \times Dx^2$; $Dx = D1 - (D1 - D2)/L \times x$

  Let $(D1-D2)/L = k$; Then $Dx = D1 - kx$

  $d(\Delta Lx) = 4 \times P \times dx / (\pi \times (D1-kx)^2 \times E)$

  Integrating from x=0 to x=L 4PL/(πED1D2)

$$\int_0^L d(\Delta x) = \int_0^L 4 \times P \times dx / (\pi \times (D1-kx)^2 \times E)$$

  Let $D1 - kx = \lambda$; then $dx = -(d\lambda/k)$

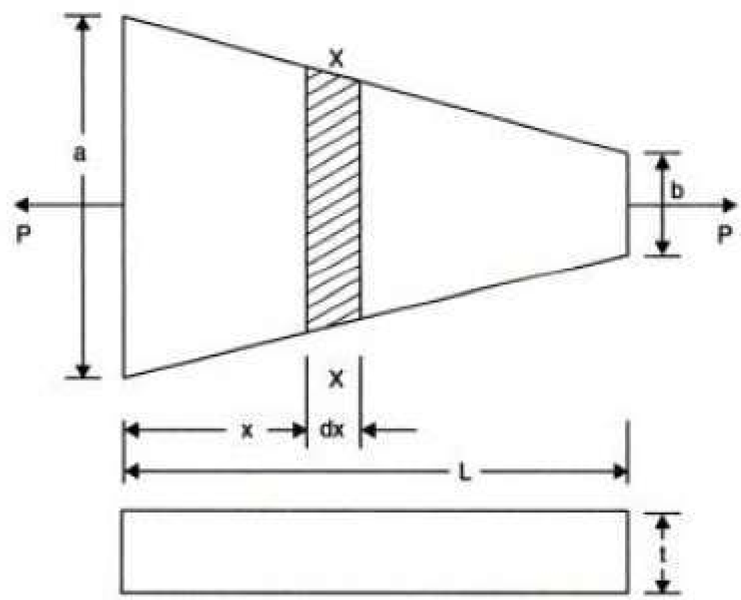
  When x=0, λ=D1; When x=L, λ=D2

$$\int_0^L d(\Delta Lx) = \int_{D1}^{D2} -4 \times P \times dx / (\pi \times \lambda^2 k \times E)$$

$$\Delta Lx = 4PL/(\pi ED1D2)$$

1.12 ANALYSIS OF UNIFORMLY TAPERING RECTANGULAR BAR

A bar of constant thickness and uniformly tapering in width from one end to the other end is shown in Fig. 1.14.

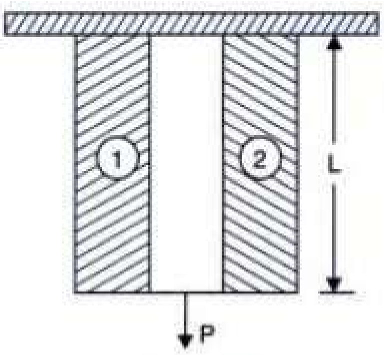


Let P = Axial load on the bar
L = Length of bar
a = Width at bigger end
b = Width at smaller end
E = Young's modulus
t = Thickness of bar

$$dL = \frac{PL}{Et(a-b)} \log_e \frac{a}{b} .$$

1.13 ANALYSIS OF BARS OF COMPOSITE SECTIONS

- A bar, made up of two or more bars of equal lengths but of different materials rigidly fixed with each other and behaving as one unit for elongation and shortening when subjected to axial loads is called composite bar.
- Consider a composite bar as showı



- Let

  P→ Applied load

  L→ Length of bar

  A1→ Area of cross section of Inner member

  A2→ Cross sectional area of Outer member

1.13 ANALYSIS OF BARS OF COMPOSITE SECTIONS

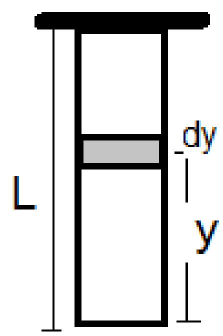
- Strain developed in the outer member= Strain developed in the inner member

  $ς1/E1 = ς2/E2$

- Total load (P)= Load in the inner member (P1) + Load in the outer member (P2)
- $ς1 \times A1 + ς2 \times A2 = P$
- Solving above two equations, we get the values of ς1, ς2 & e1 and e2

PRODUCED IN A BAR DUE TO ITS SELF WEIGHT

- Consider a bar of length L, area of cross section A rigidly fixed at one end. Let ρ be the density of the material. Consider an infinitesimal element of thickness (ce y from the bottom of the bar.



- The force acting on the element considered= weight of the portion below it=ρAgy

PRODUCED IN A BAR DUE TO ITS SELF WEIGHT

- Tensile stress developed= Force acting on the element/Area of cross section= ρgy.

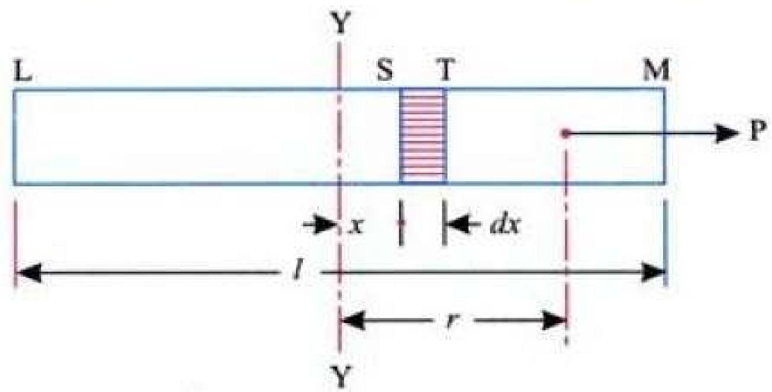
- From the above equation, it is clear that the maximum stress at the section where y=L, ie at the fixed end (ρgL) and minimum stress is at the free end(=0)

$$\Delta L_y = \int_0^L \rho g y\,dy / AE = \rho g L^2 / 2AE$$

- Elongation due to self weight

1.15 STRESS IN BAR DUE TO ROTATION

Consider a bar of length l rotating about the axis y at a constant angular velocity ω. Consider an infinitesimal element of thickness dx at a distance x from the axis of rotation.



Tensile force on element ST= Centrifugal force on element TM

Centrifugal force on element TM= Mass of element TM x r x ω^2= $\{l/2 - (x+dx)\}$ x A x ρ x r x ω^2

r= x + ½ x (l/2- (x+dx)

As dx is numerically very small, x + dx≈x

Hence tensile force on element ST=(l/2 – x) x A x{x + ½ x (l/2-x)} x ρ x ω^2

= A x ρ x ω^2 x $(l^2/4 - x^2)/2$

1.15 STRESS IN BAR DUE TO ROTATION

Tensile stress developed= Tensile force/cross sectional area= $A \times \rho \times \omega^2 \times (l^2/4 - x^2)/2A$

$\sigma rod = \rho \times \omega^2 \times (l^2/4 - x^2)/2$

$\sigma rod = 0$, when $x = l/2$

σrod =Maximum when $d(\sigma rod)/dx = 0$; ie when $x = 0$

$\sigma rodmax == \rho \times \omega^2 \times l^2/8$

Extension of element= $\sigma rod \times dx/E$

Extension of entire bar= $\int_0^l \rho \times \omega^2 \times (l^2/4 - x^2)dx/2 = \rho \times \omega^2 \times l^3/12E$

Extension of entire bar= $\rho \times \omega^2 \times l^3/12E$

1.16 THERMAL STRESS

- Thermal stresses are the stresses induced in a body due to change in temperature. Thermal stresses are set up in a body, when the temperature of the body is raised or lowered and the body is restricted from expanding or contracting
- Consider a body which is heated to a certain temperature

  Let

  L= Original length of the body

  Δ T=Rise in temp

  E=Young's modulus

  α=Coefficient of linear expansion

  dL= Extension of rod due to rise of temp
- If the rod is free to expand, Thermal strain developed

  $et = \Delta L/L = \alpha \times \Delta T$

1.16 THERMAL STRESS

- The extension of the rod, $\Delta L = L \times \alpha \times \Delta T$
- If the body is restricted from expanding freely, Thermal stress developed is $\varsigma t/et = E$
- $\varsigma t = E \times \alpha \times \Delta T$
- Stress and strain when the support yields:-
  If the supports yield by an amount equal to δ, then the actual expansion is given by the
  difference between the thermal strain and δ
  Actual strain, $e = (L \times \alpha \times \Delta T - \delta)/L$
  Actual stress= Actual strain x $E = (L \times \alpha \times \Delta T - \delta)/L \times E$

UNIT II

SHEAR AND BENDING IN BEAMS

APPLIED AND REACTIVE FORCES

- Forces that act on a Body can be divided into two Primary types: applied and reactive.
- In common Engineering usage, applied forces are forces that act directly on a structure like, dead, live load etc.)
- Reactive forces are forces generated by the action of one body on another and hence typically occur at connections or supports.
- The existence of reactive forces follows from Newton's third law, which state that to every action , there is an equal and opposite reaction.

SUPPORTS

To bear or hold up (a load, mass, structure, part, etc.); serve as a foundation or base for any structure.

To sustain or withstand (weight, pressure, strain, etc.) without giving way

It is a aid or assistance to any structure by preserve its load

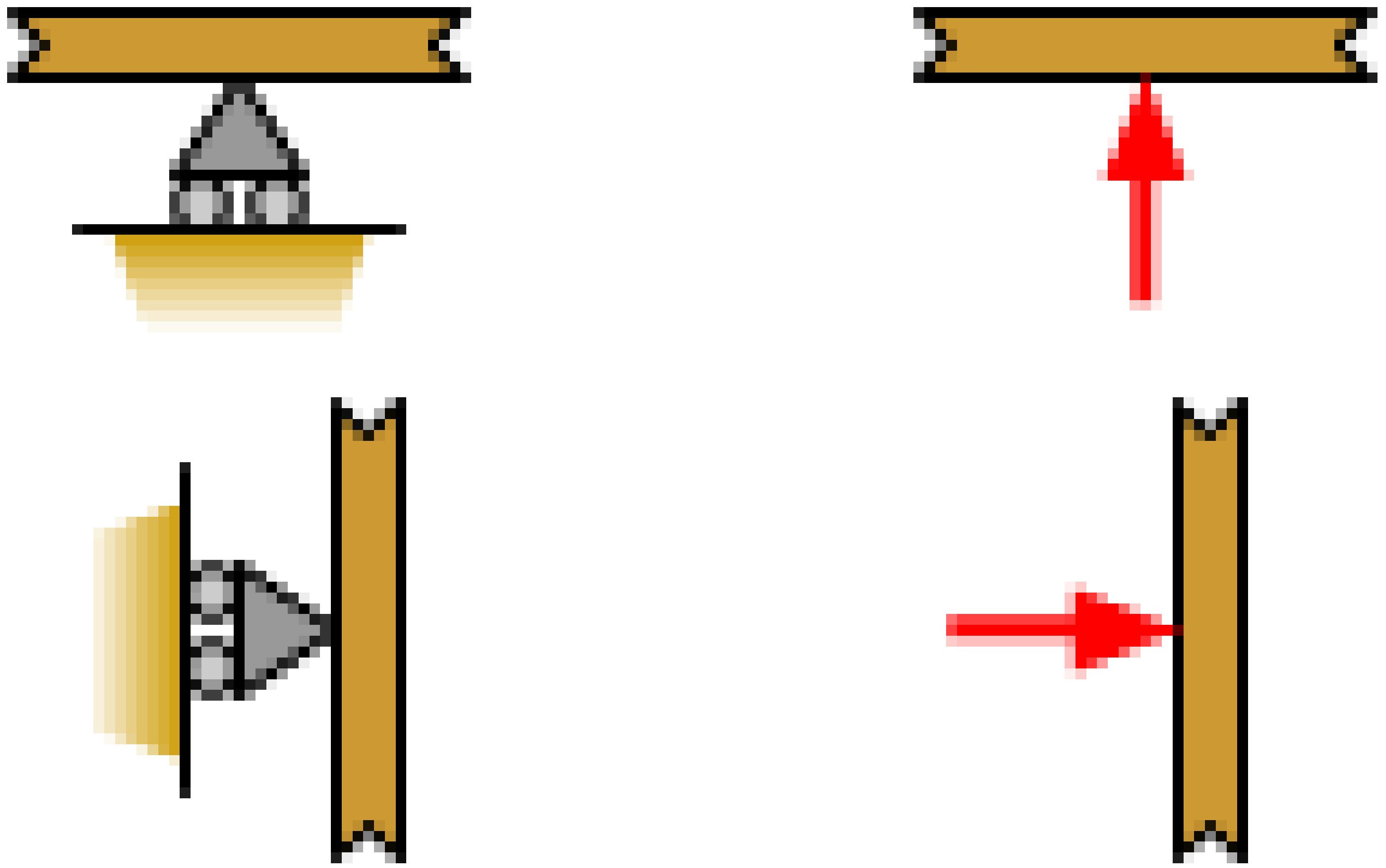
Supports are used to connect structures to the ground or other bodies in order to restrict (confine) their movements under the applied loads. The loads tend to move the structures, but supports prevent the movements by exerting opposing forces, or reactions, to neutralize the effects of loads thereby keeping the structures in equilibrium.

TYPES OF SUPPORTS

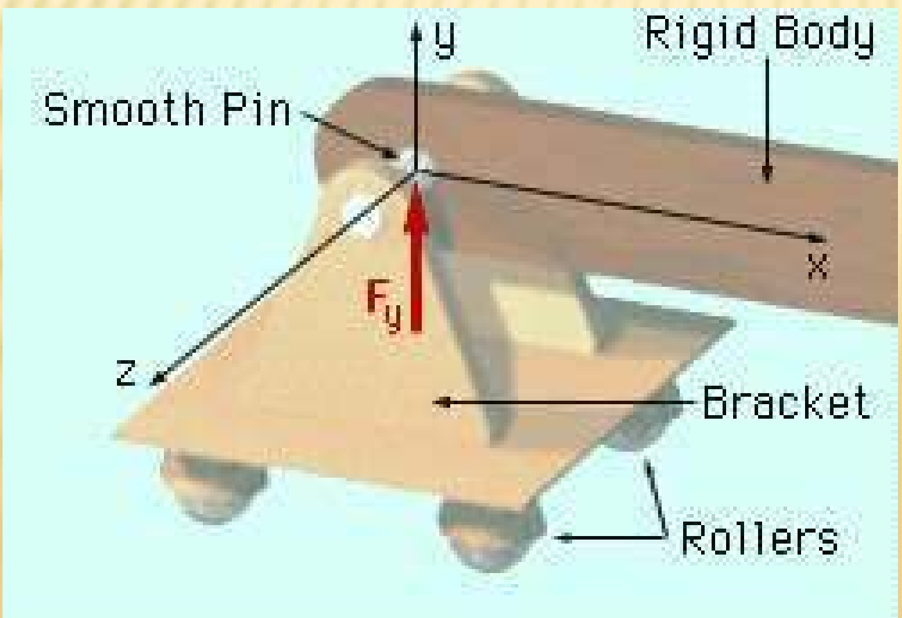
- Supports are grouped into three categories, depending on the number of reactions (1,2,or3) they exert on the structures.
- 1) Roller support
- 2) Hinge support
- 3) fixed support

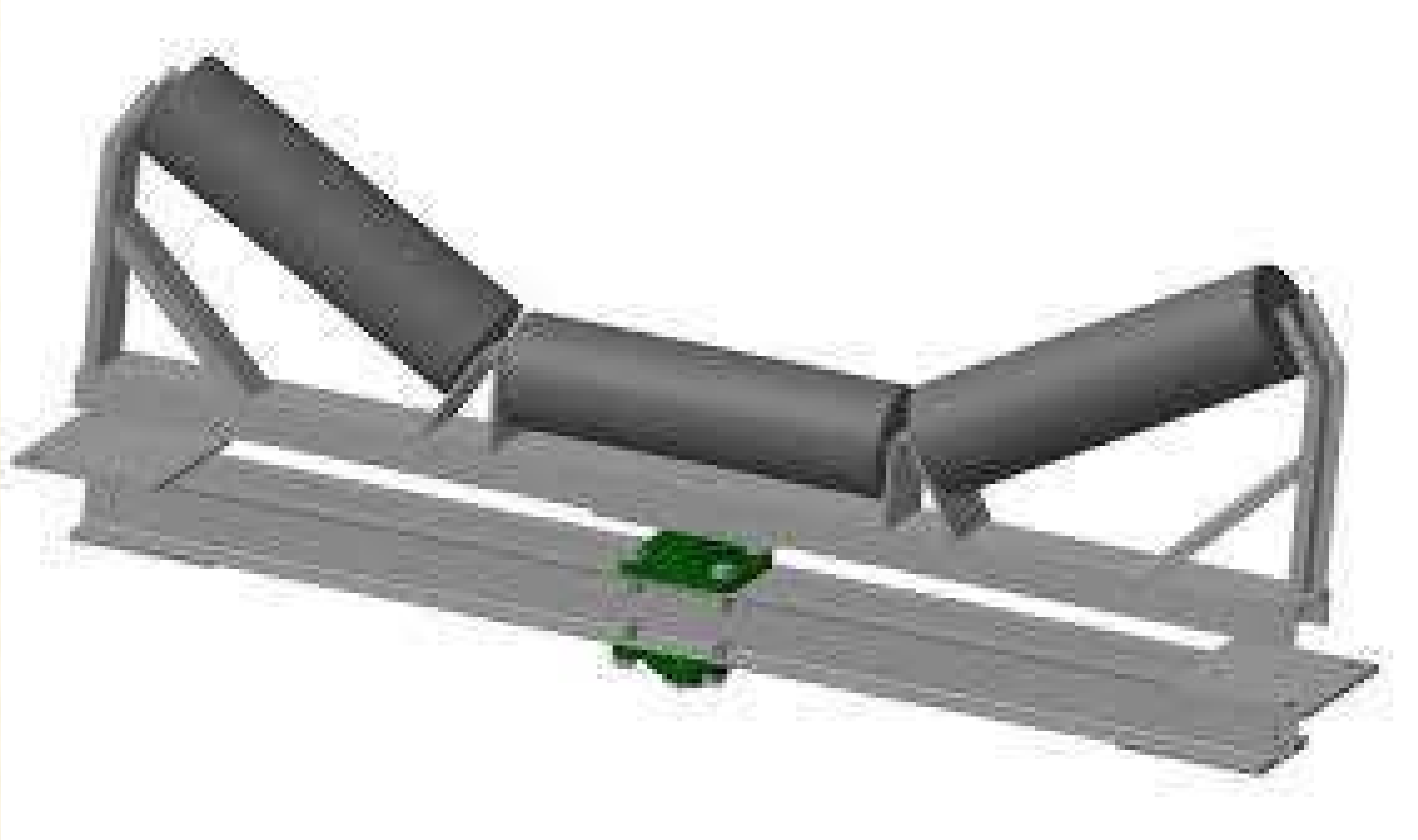
ROLLER SUPPORT

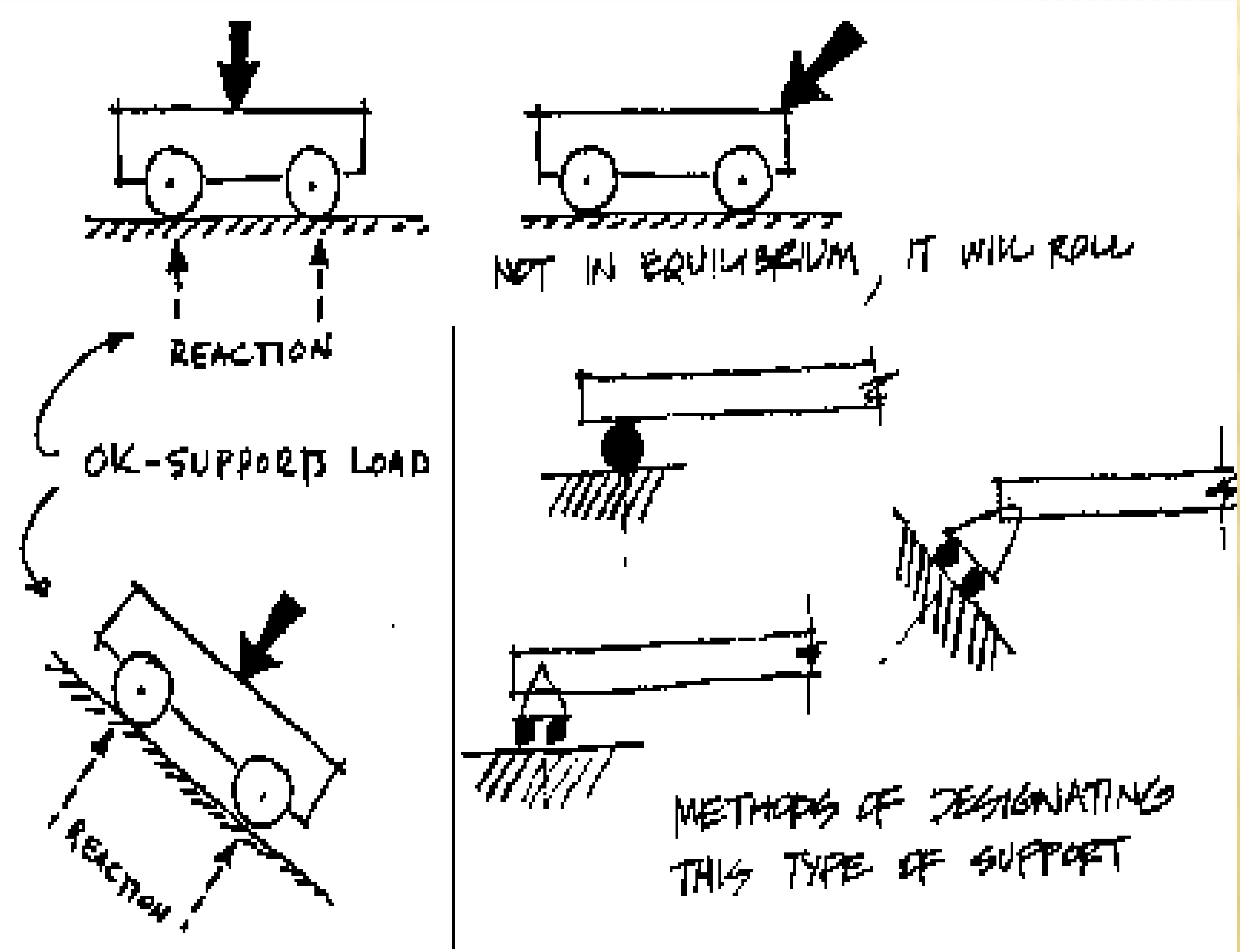
- Roller supports are free to rotate and translate along the surface upon which the roller rests.
- The surface can be horizontal, vertical, or sloped at any angle.
- The resulting reaction force is always a single force that is perpendicular to, and away from, the surface



Restrains the structure from moving in one or two perpendicular directions.


y
Rigid Body
Smooth Pin
x
F_y
z
Bracket
Rollers

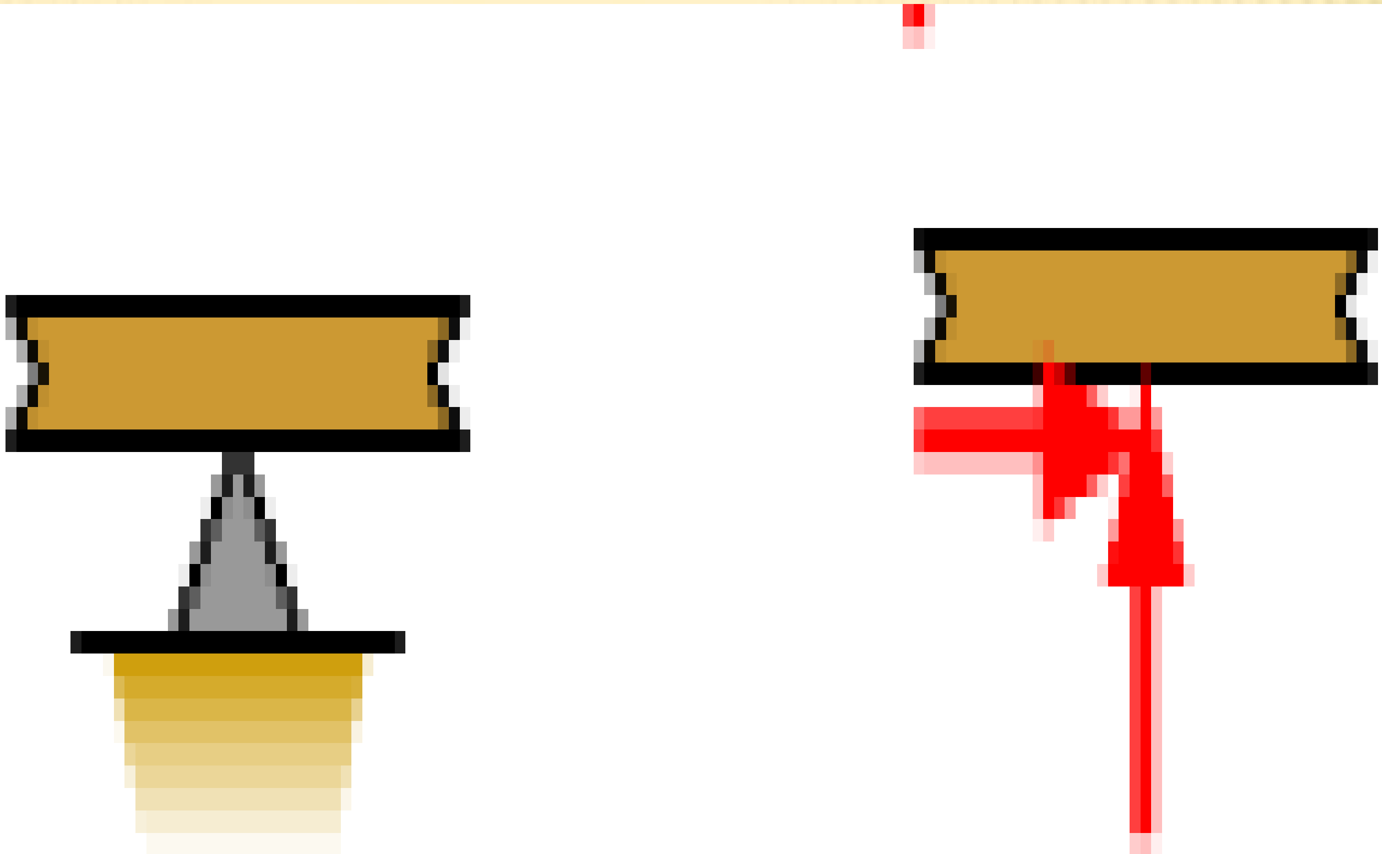


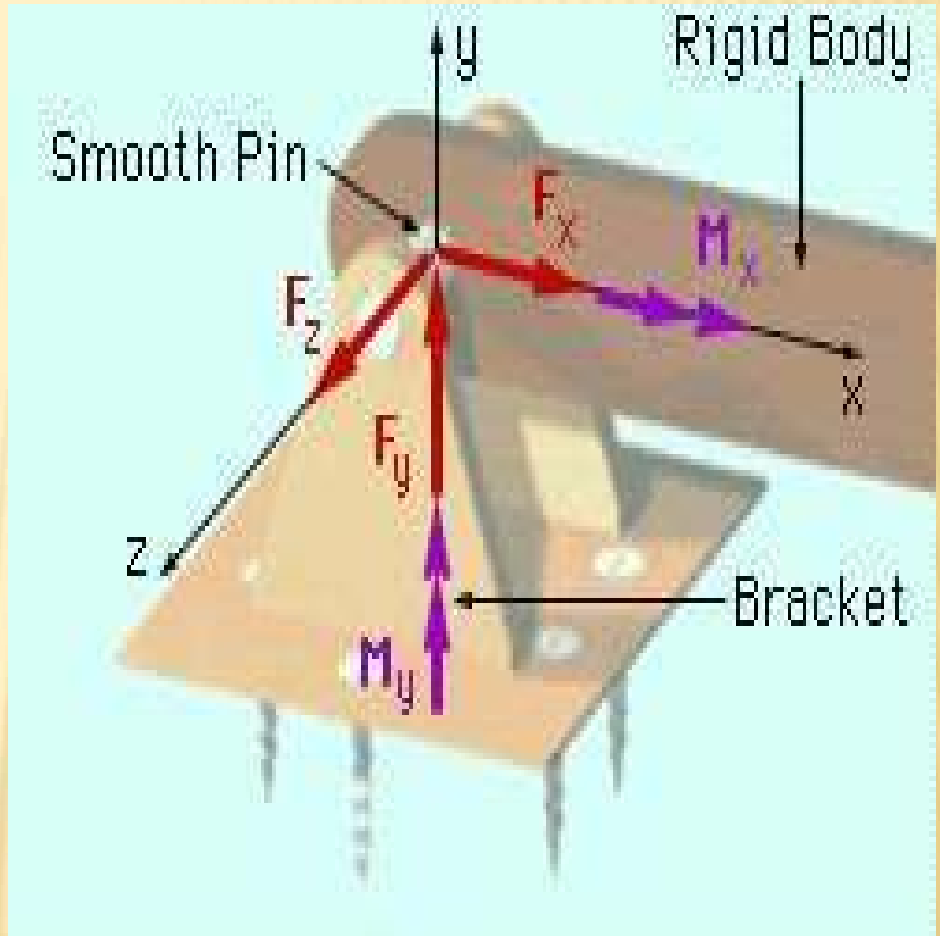


REACTION
OK-SUPPORTS LOAD
REACTION
NOT IN EQUILIBRIUM, IT WILL ROLL
METHODS OF DESIGNATING
THIS TYPE OF SUPPORT

HINGE SUPPORT

- A Hinge support can resist both vertical and horizontal forces but not a moment. They will allow the structural member to rotate, but not to translate in any direction
- Pin or hinge support is used when we need to prevent the structure from moving or restrain its translational degrees of freedom.
- A **hinge** is a type of bearing that connects two solid objects, typically allowing only a limited angle of rotation between them. Two objects connected by an ideal hinge rotate relative to each other about a fixed axis of rotation.




y
Rigid Body
Smooth Pin
F_x
M_x
F_z
x
F_y
z
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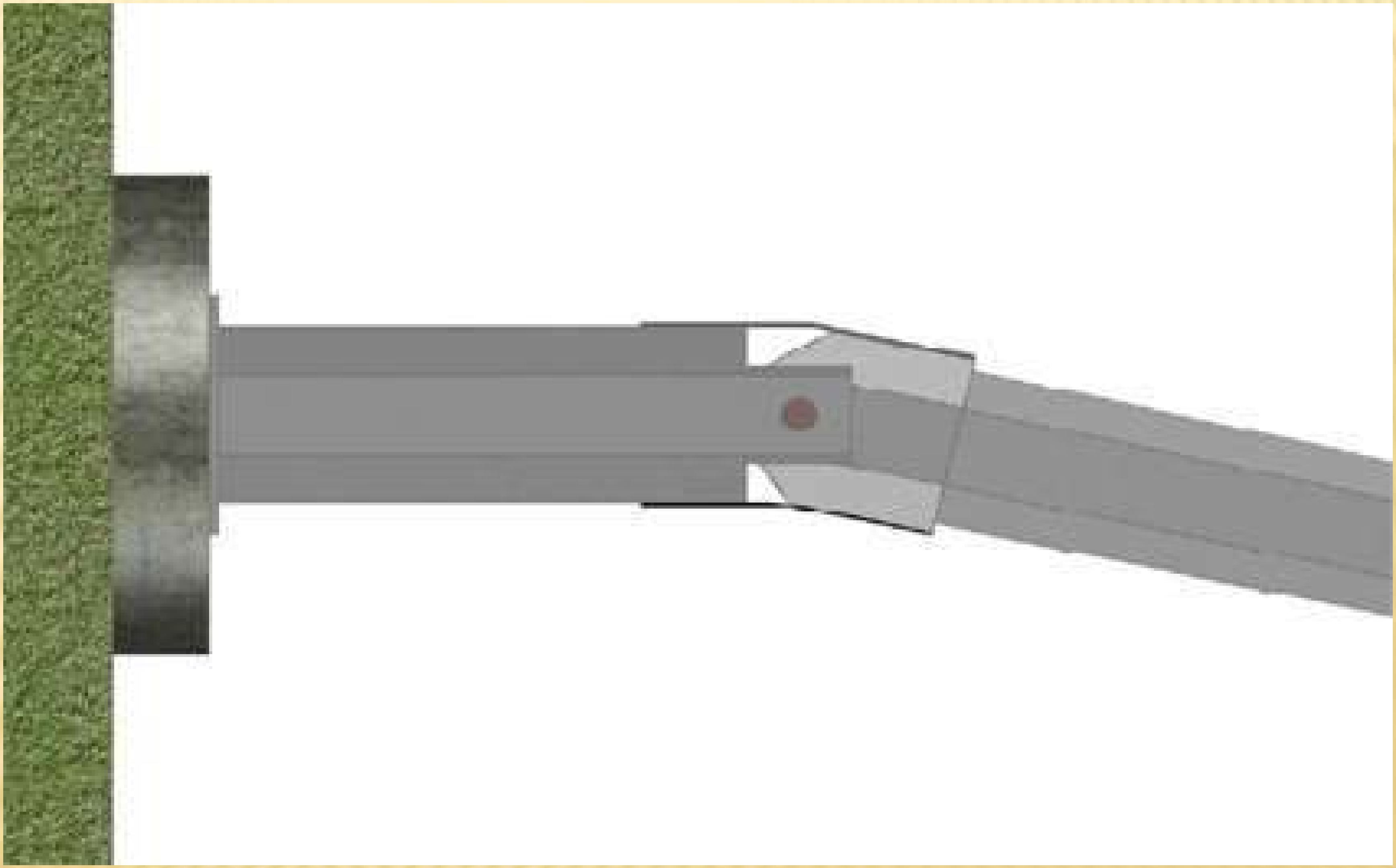
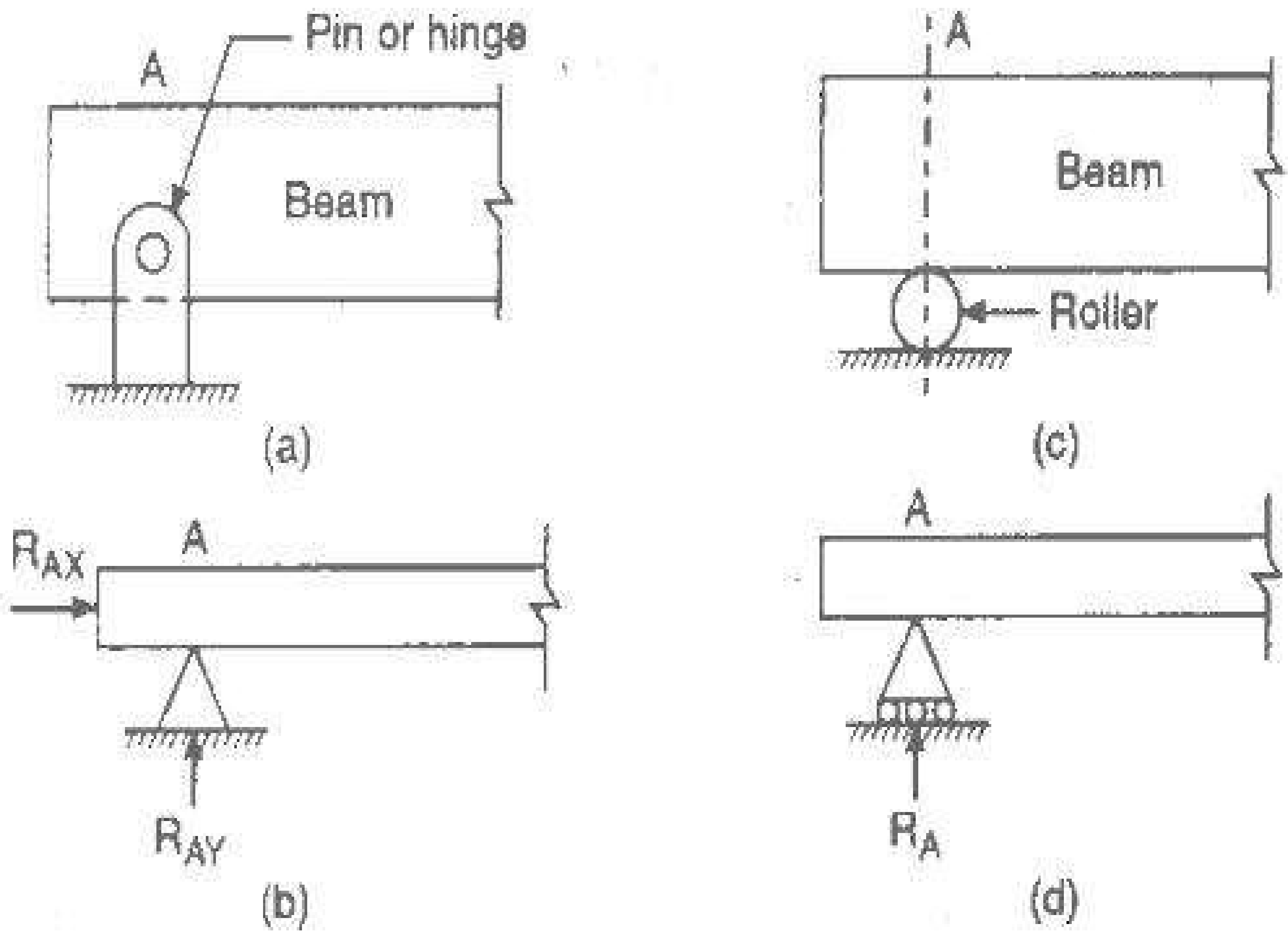
M_y

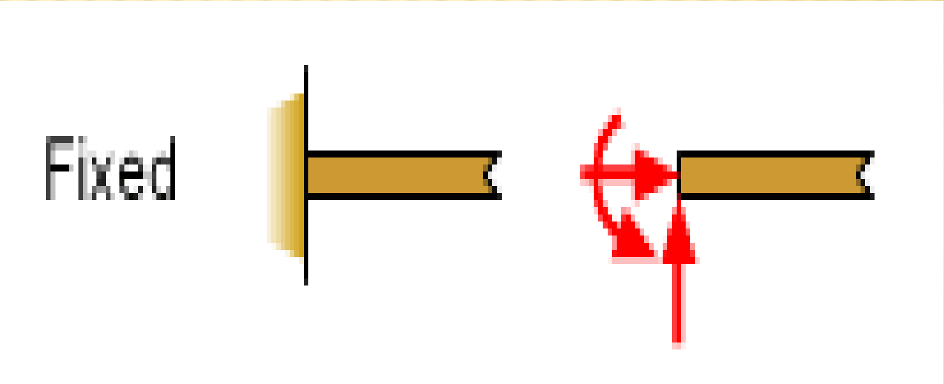


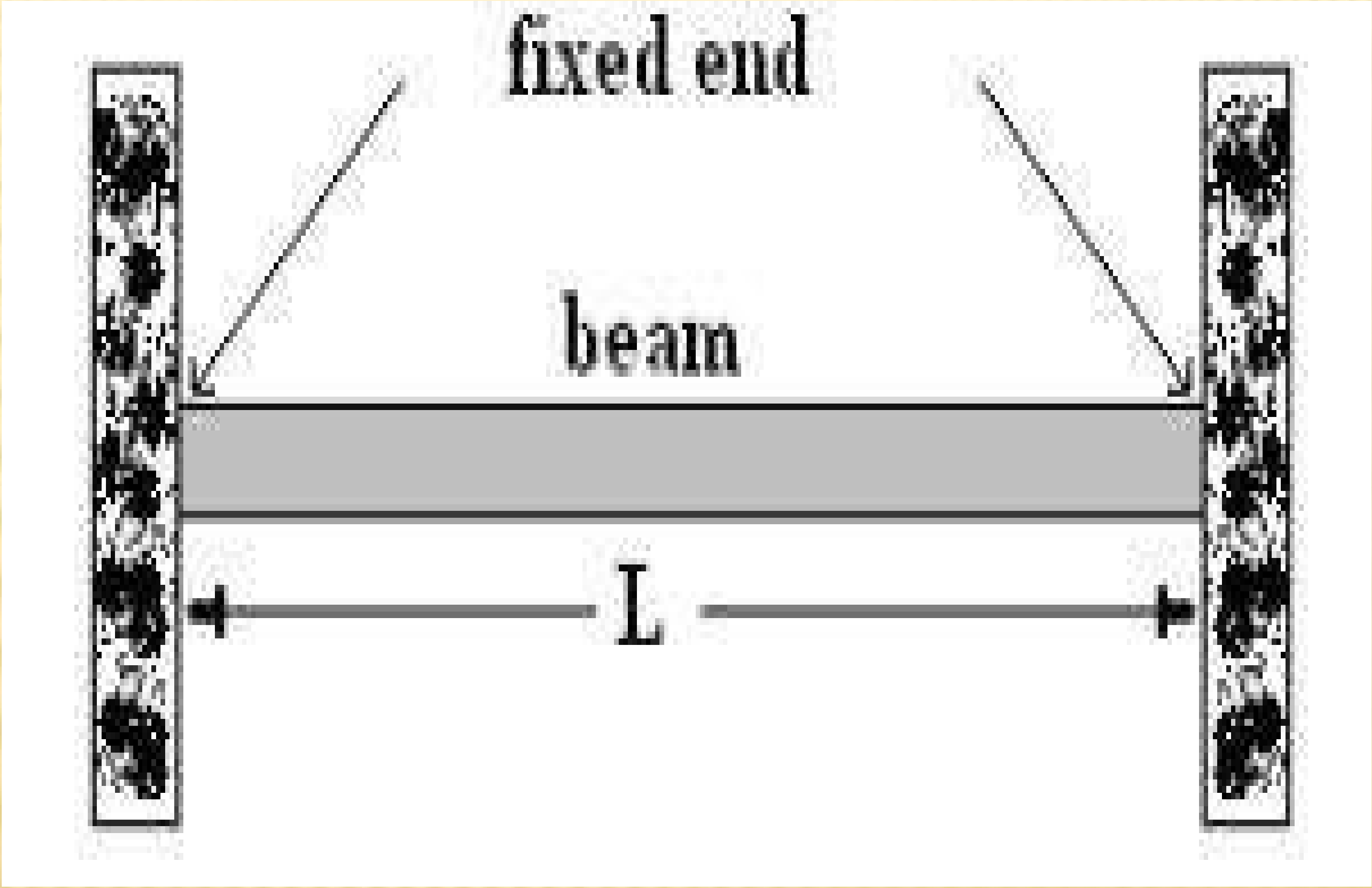


Pin or hinge
A
Beam
(a)
A
Beam
Roller
(c)
R_{AX}
A
R_{AY}
(b)
A
R_A
(d)

FIXED SUPPORT

- Fixed supports can resist vertical and horizontal forces as well as a moment. Since they restrain both rotation and translation, they are also known as rigid supports.



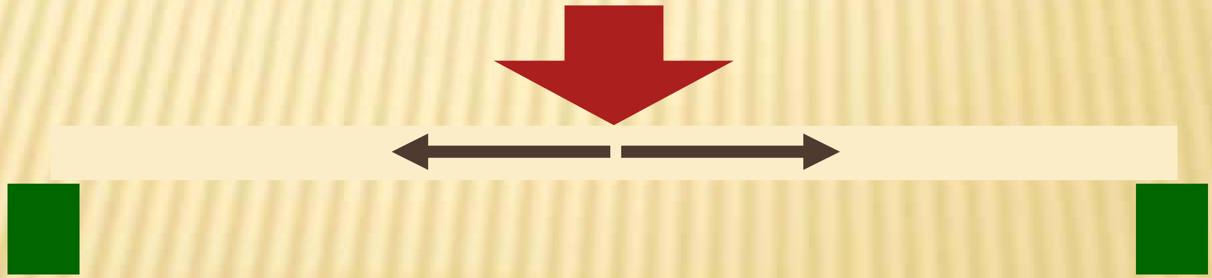

fixed end
beam
L

BEAM

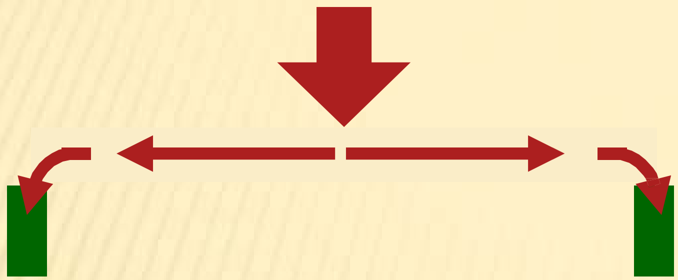
- A beam is a structural member(horizontal) that is design to support the applied load (vertical). It resists the applied loading by a combination of internal transverse **shear force** and bending **moment**.
- It is perhaps the most important and widely used structural members and can be classified according to its support conditions.

Beams

- Extremely common structural element
- In buildings majority of loads are vertical and majority of useable surfaces are horizontal



Beams



devices for transferring vertical loads horizontally

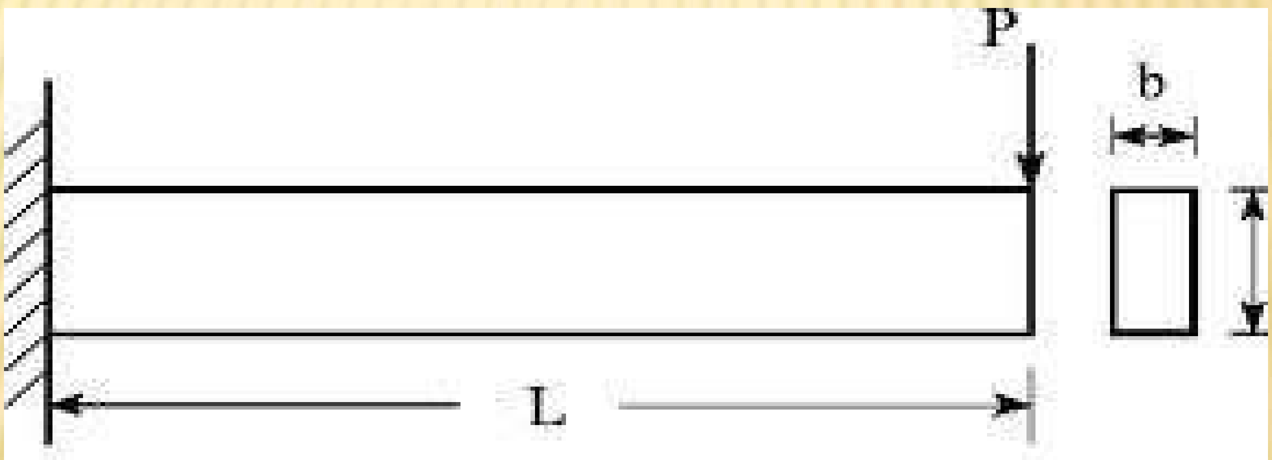
action of beams involves combination of bending and shear

TYPES OF BEAMS

- The following are the important types of beams:
- 1. Cantilever
- 2. simply supported
- 3. overhanging
- 4. Fixed beams
- 5. Continuous beam

CANTILEVER BEAM

- A beam which is fixed at one end and free at the other end is known as cantilever beam.



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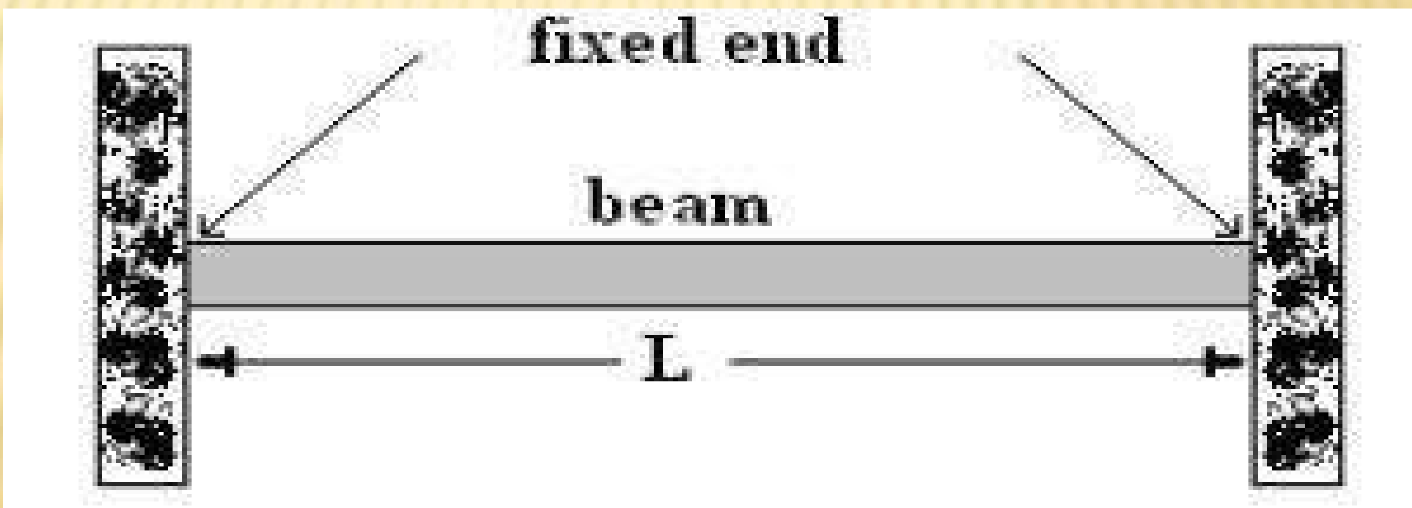
SIMPLY SUPPORTED BEAMS

- A beam supported or resting freely on the supports at its both ends,

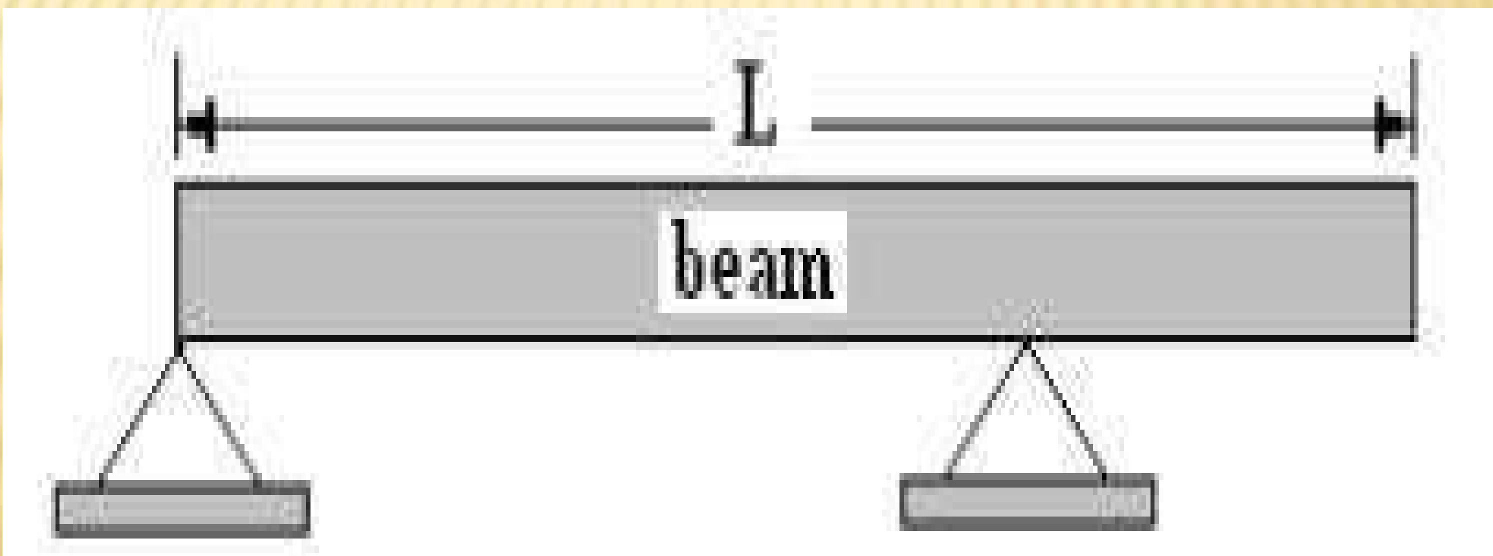
FIXED BEAMS

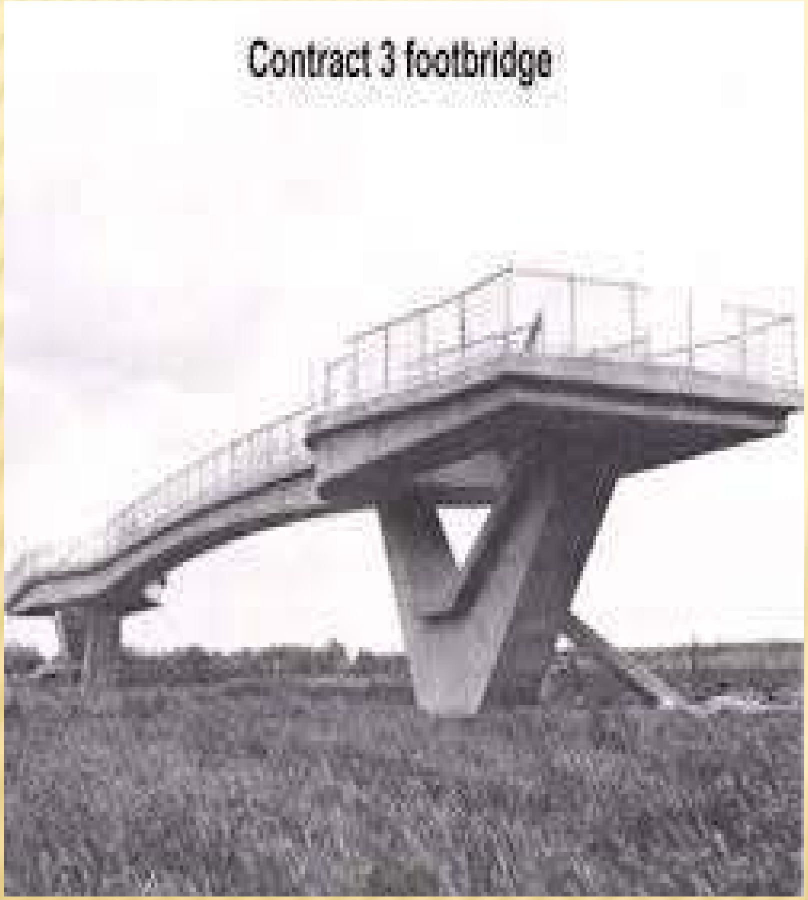
- A beam whose both ends are fixed and is restrained against rotation and vertical movement. Also known as built-in beam or encastred beam.



OVERHANGING BEAM

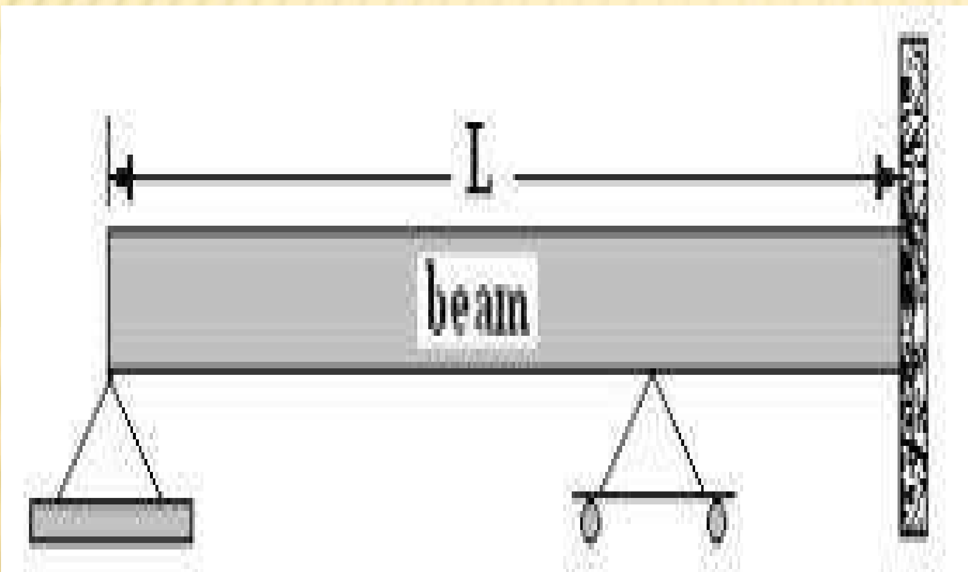
- If the end portion of a beam is extended outside the supports.




Contract 3 footbridge

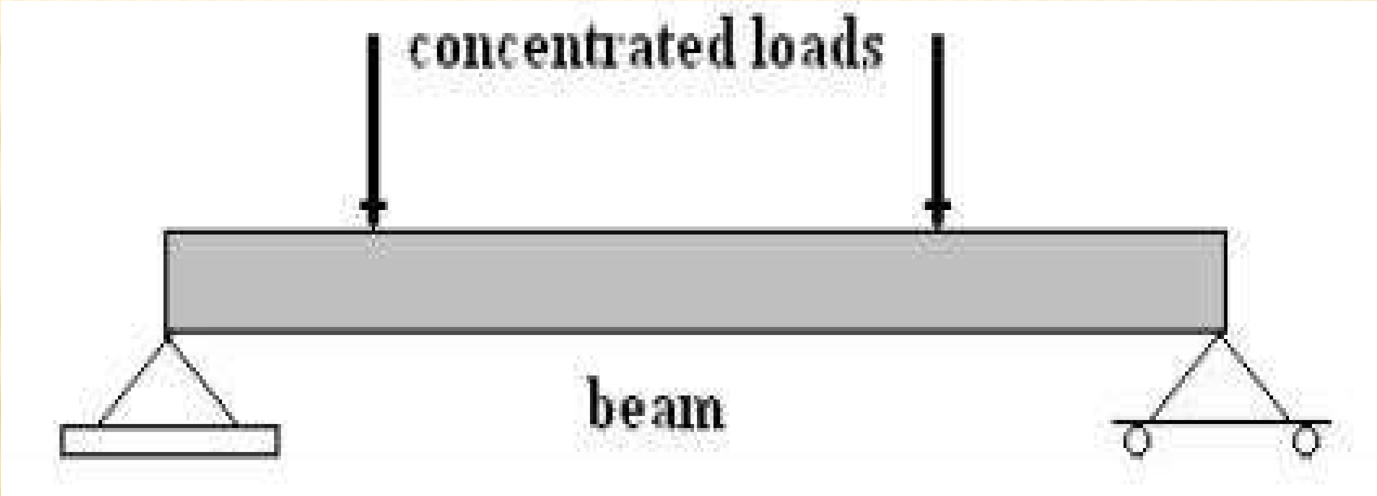
CONTINUOUS BEAMS

- Abeam which is provided with more than two supports.



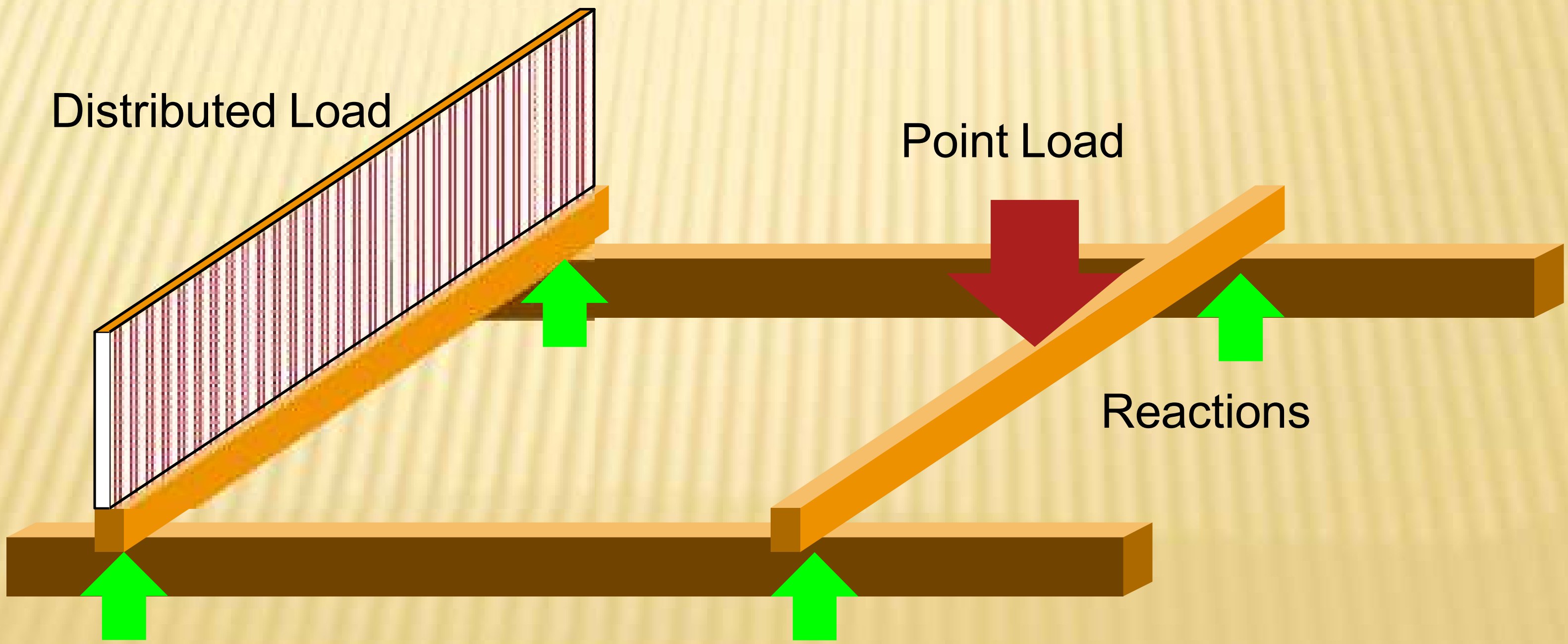
TYPES OF LOADS

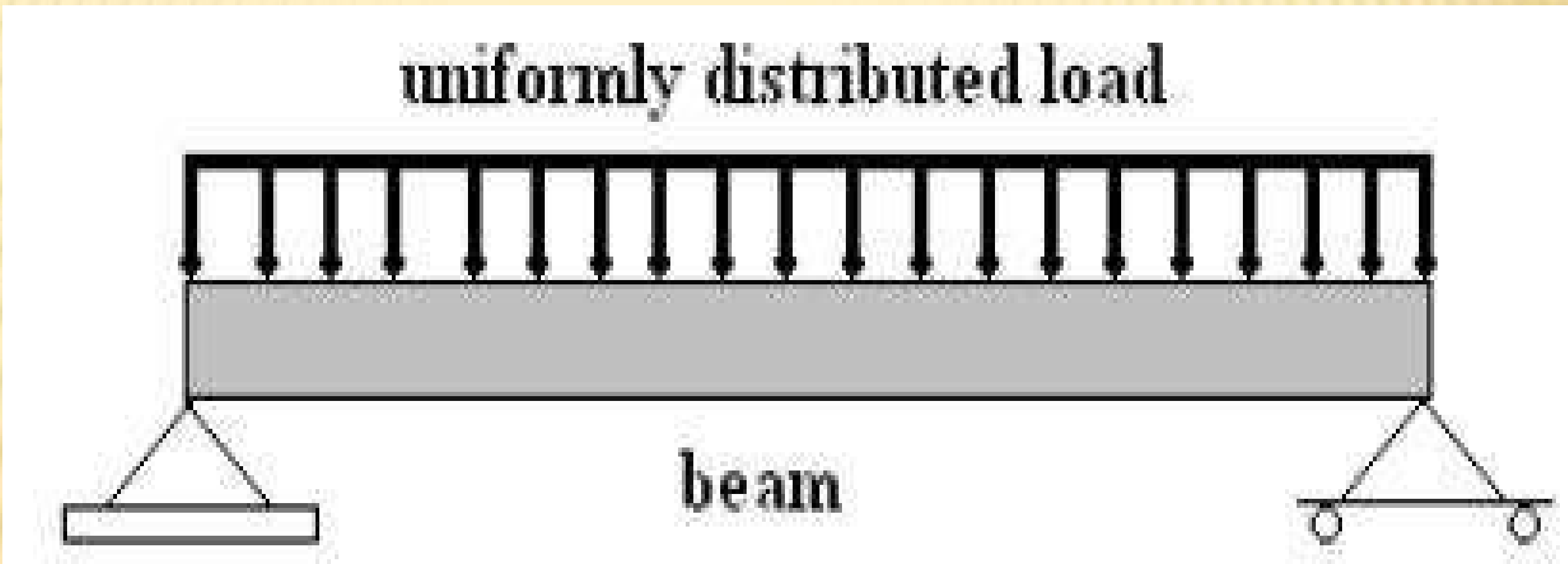
- Concentrated load assumed to act at a point and immediately introduce an oversimplification since all practical loading system must be applied over a finite area.

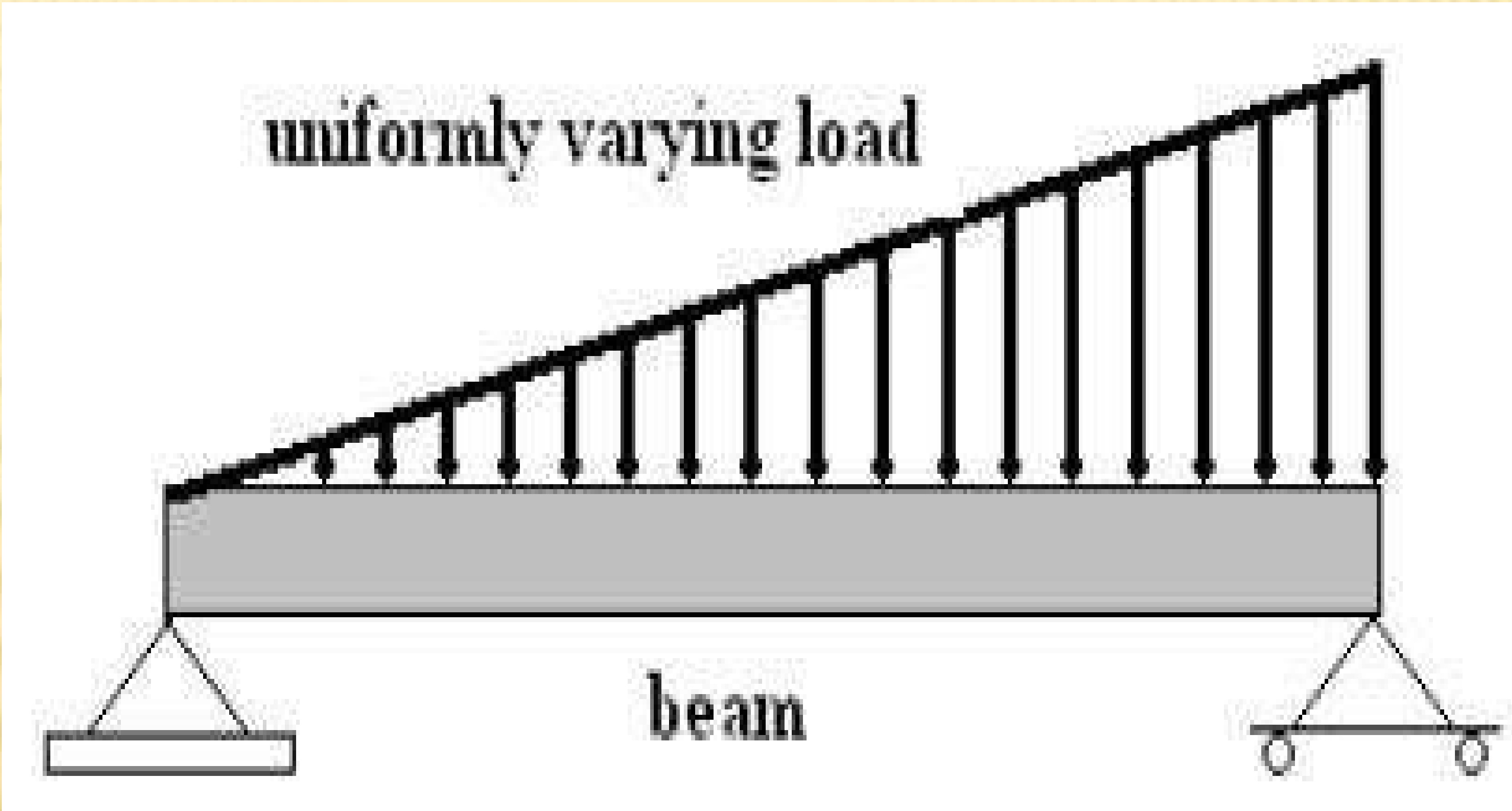


Loads on Beams

- Point loads, from concentrated loads or other beams
- Distributed loads, from anything continuous

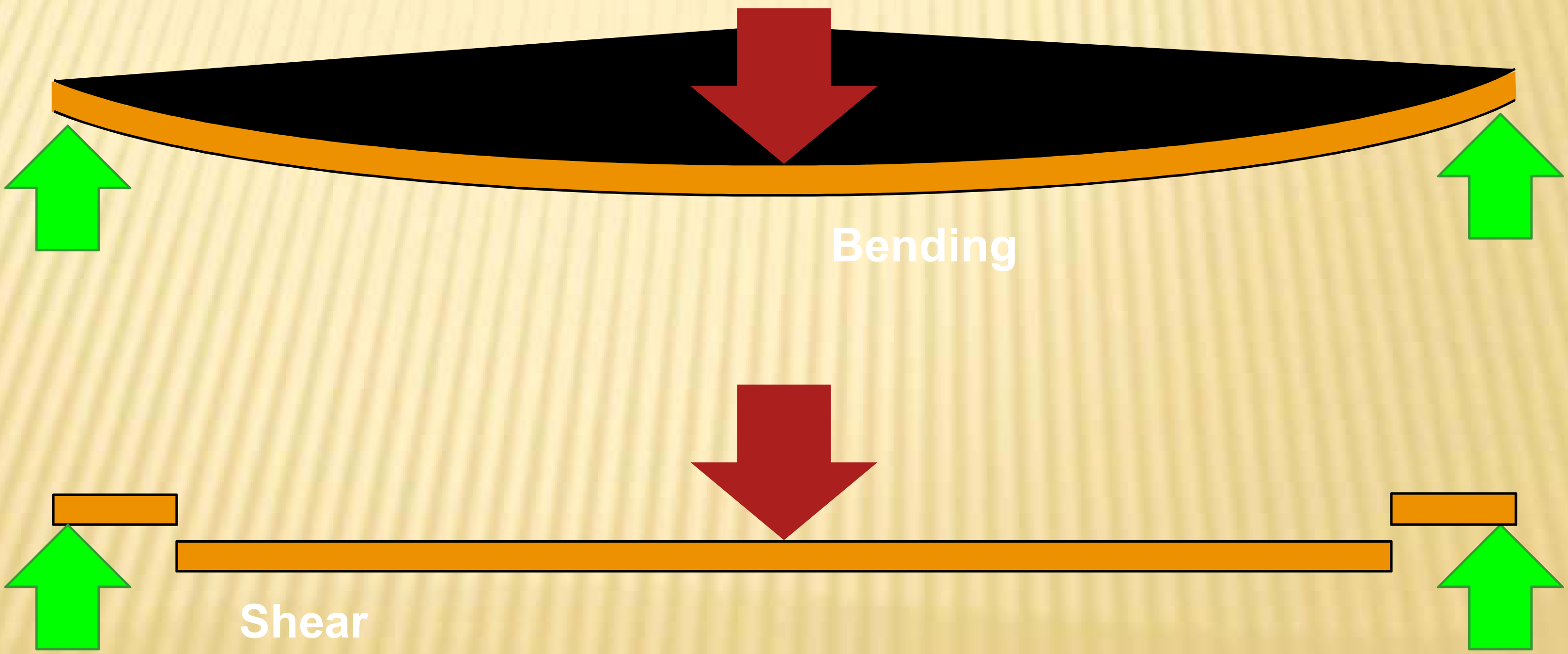



uniformly distributed load
beam

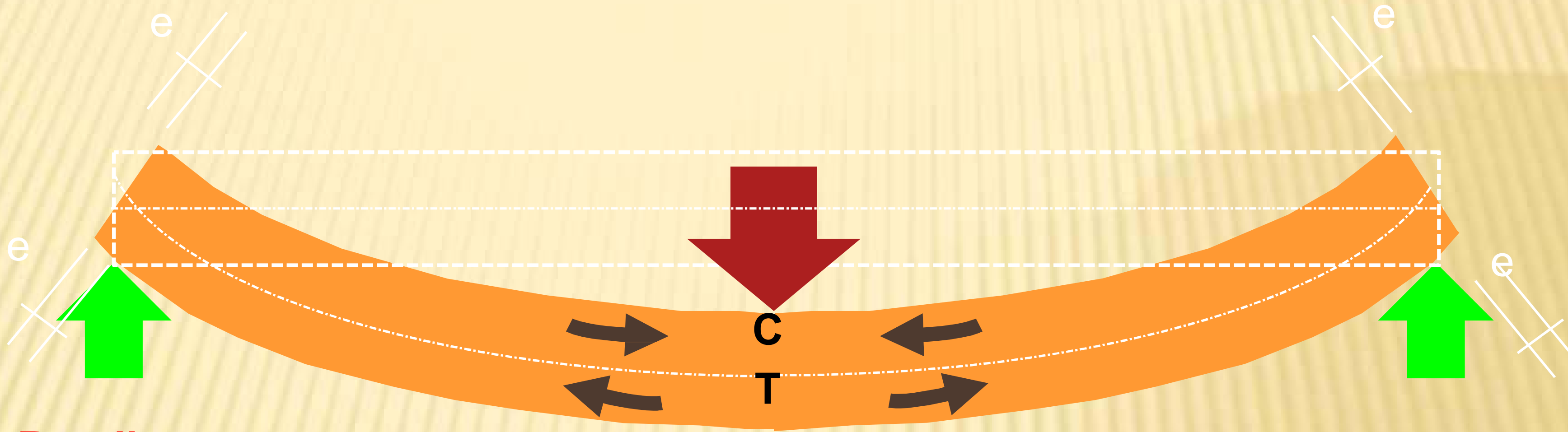

uniformly varying load
beam

What the Loads Do

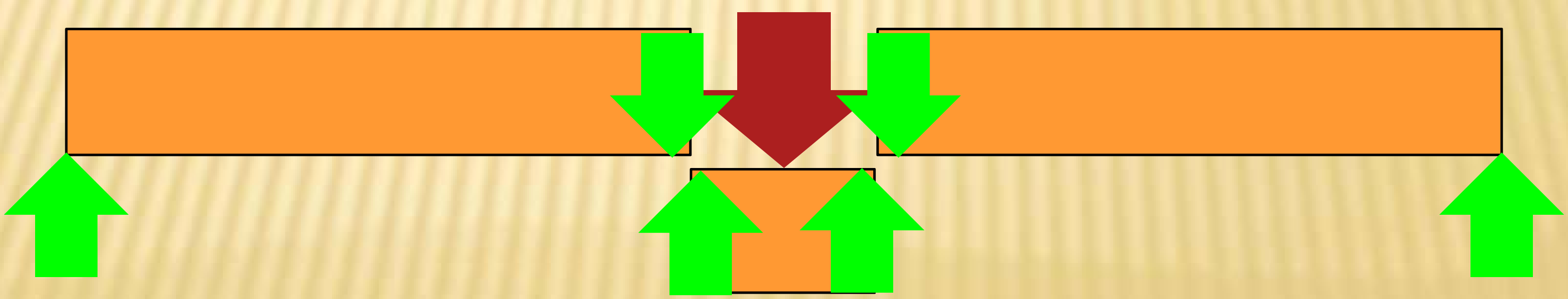
- The loads (& reactions) bend the beam, and try to shear through it



What the Loads Do



Bending



Shear

Designing Beams

- in architectural structures, bending moment more important
  - importance increases as span increases

- short span structures with heavy loads, shear dominant
  - e.g. pin connecting engine parts

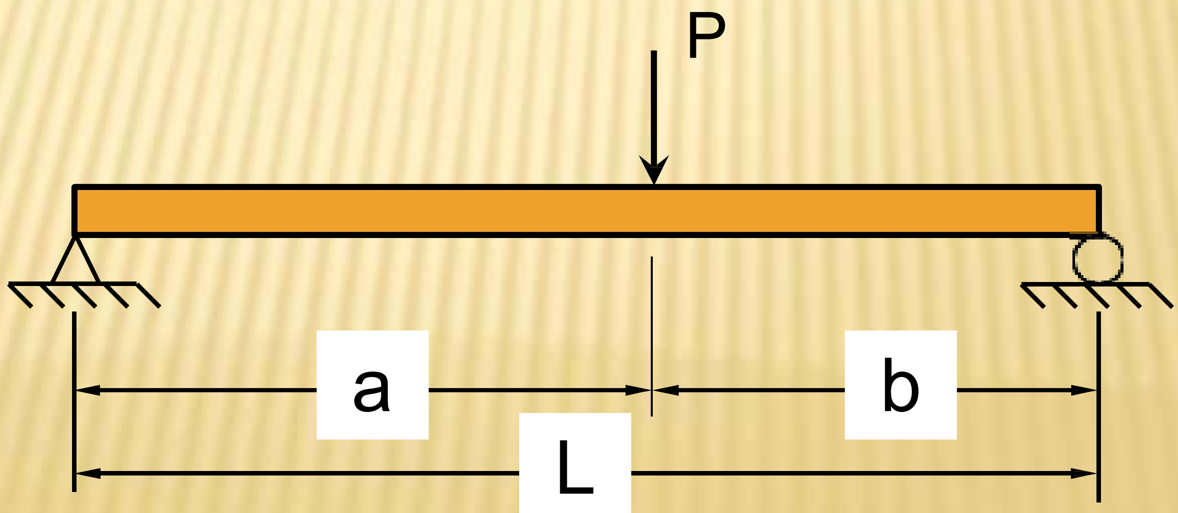
beams in building
designed for bending
checked for shear

How we calculate the Effects

- First, find ALL the forces (loads and reactions)
- Make the beam into a free body (cut it out and artificially support it)
- Find the reactions, using the conditions of equilibrium

INTERNAL REACTIONS IN BEAMS

- At any cut in a beam, there are 3 possible internal reactions required for equilibrium:
  - normal force,
  - shear force,
  - bending moment.



INTERNAL REACTIONS IN BEAMS

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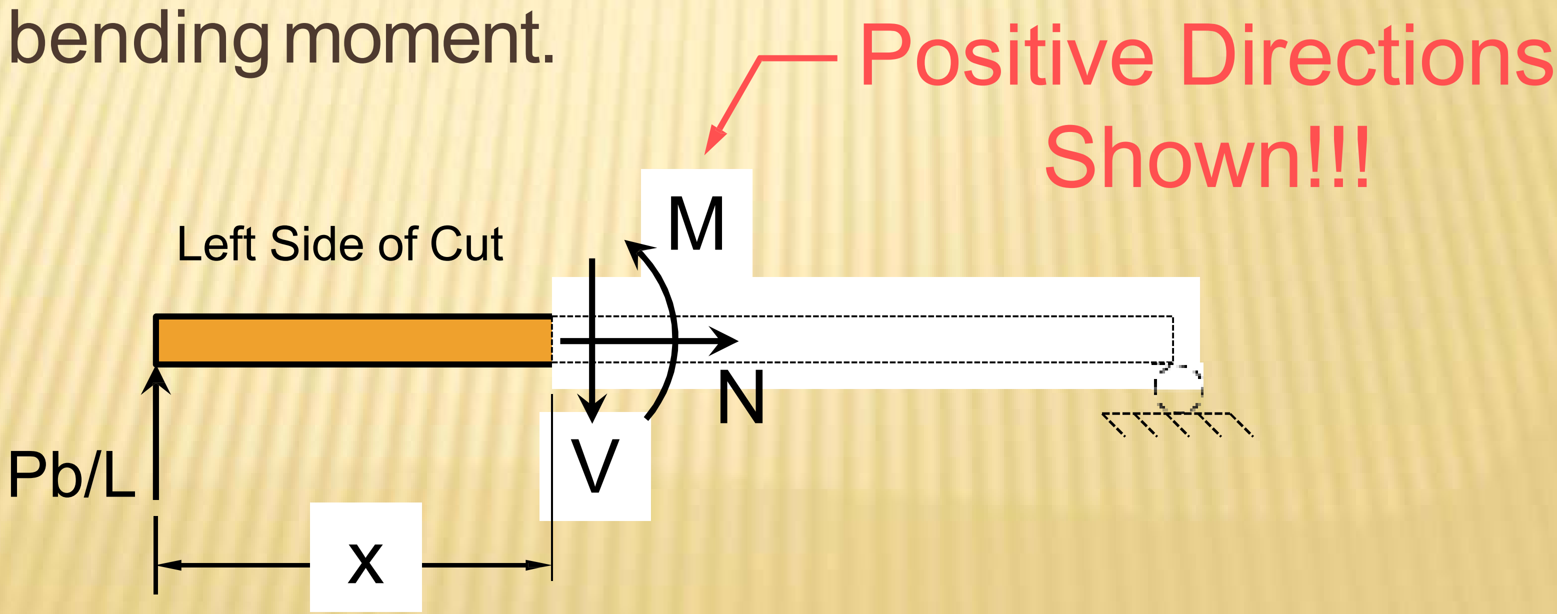

INTERNAL REACTIONS IN BEAMS

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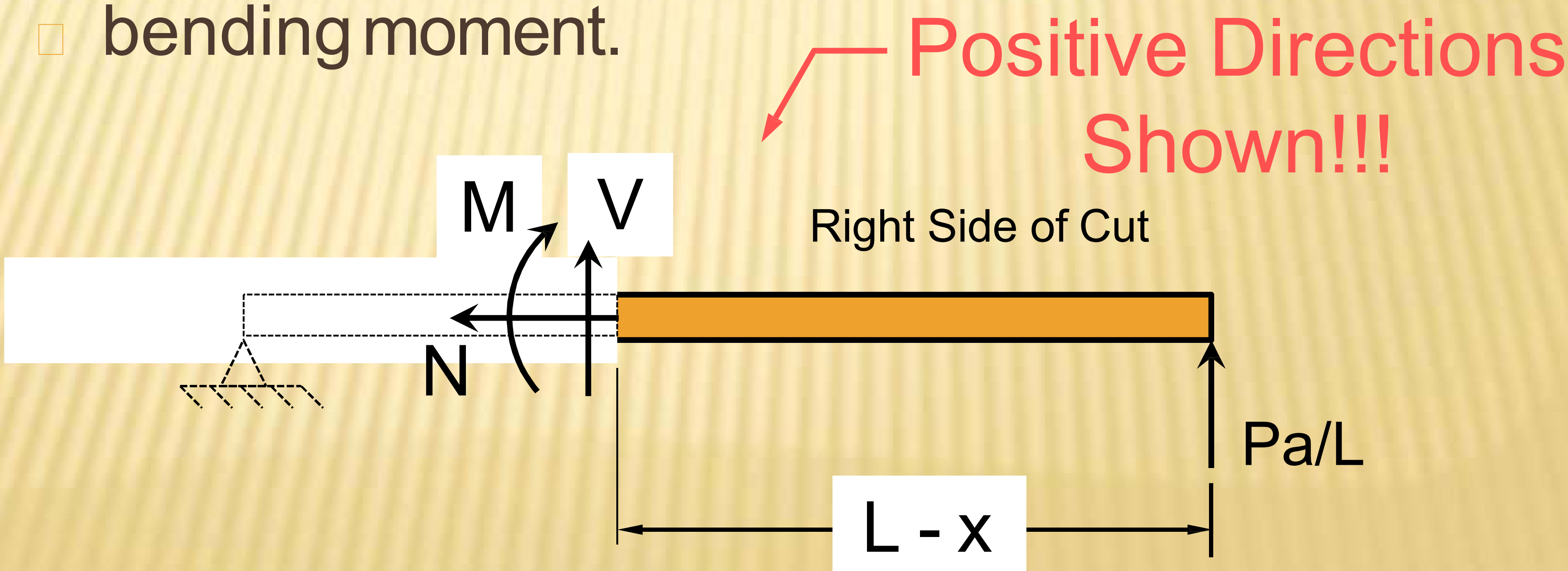
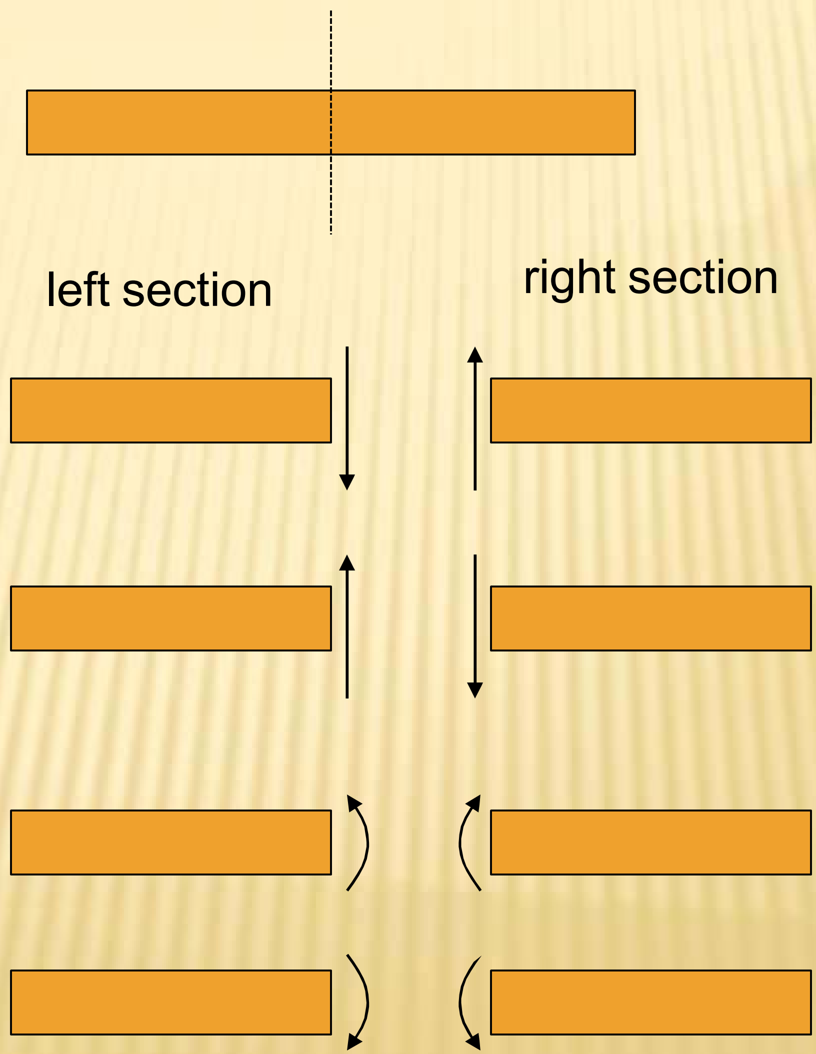


SHEAR FORCES, BENDING MOMENTS - SIGN CONVENTIONS



Shear forces:

positive shear:

negative shear:

Bending moments:

Negative moment

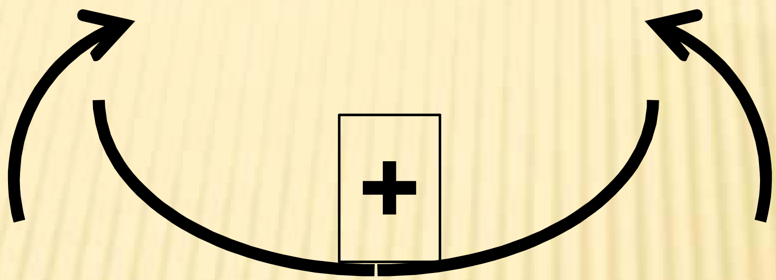
positive moment

C.W

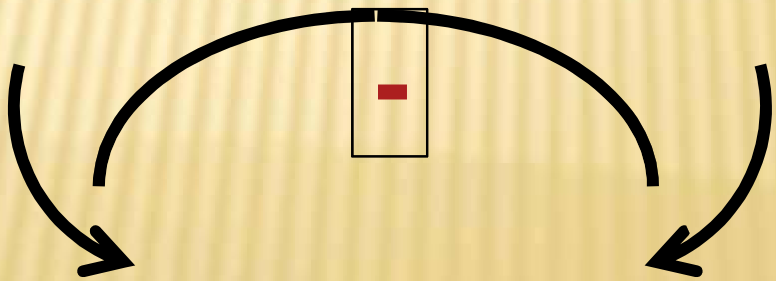
ACW

Sign Conventions
Bending Moment Diagrams (cont.)

Sagging bending moment is POSITIVE (happy)

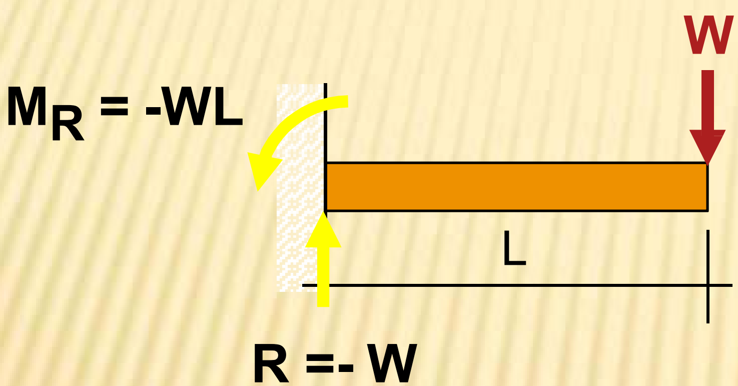


Hogging bending moment is NEGATIVE (sad)



Cantilever Beam Point Load at End

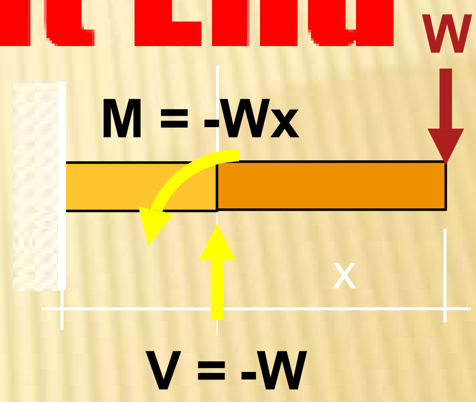
- Consider cantilever beam with point load on end



vertical reaction, R = -W
and moment reaction M_R = -WL

- **Use the free body idea to isolate part of the beam**
- **Add in forces required for equilibrium**

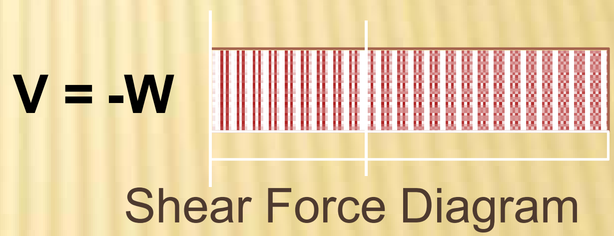
Cantilever Beam
Point Load at End



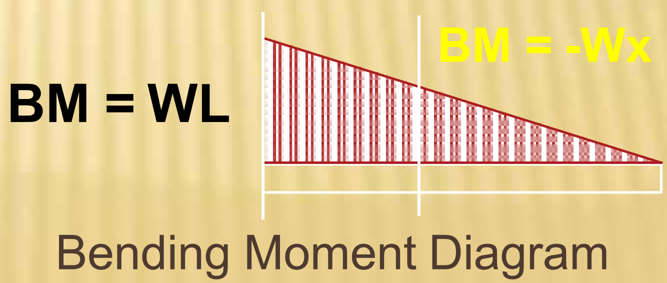
Take section anywhere at distance, x from end

Add in forces, V = -W and moment M = - Wx

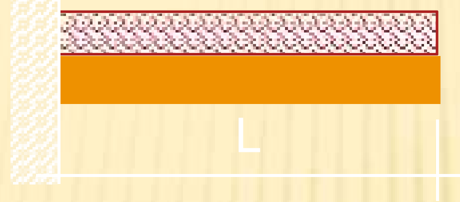
Shear V =- W constant along length



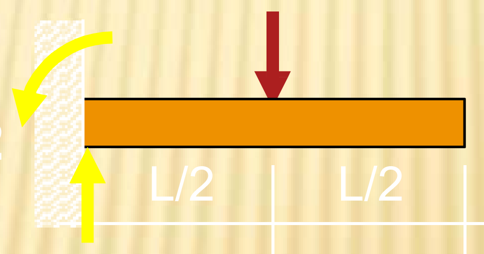
Bending Moment BM = -W.x
when x = L BM = -WL
when x = 0 BM = 0



Cantilever Beam
Uniformly Distributed Load



For maximum shear V and bending moment BM



vertical reaction, $R = W = wL$

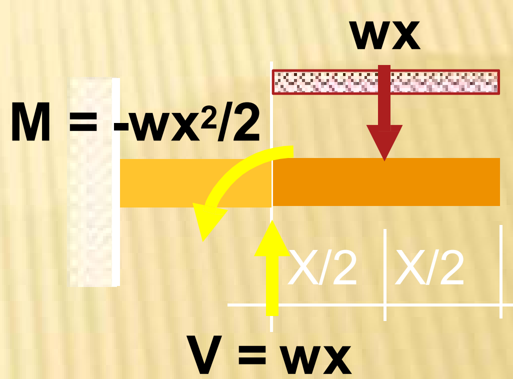
and moment reaction $M_R = -WL/2 = -wL^2/2$

Example 2 - Cantilever Beam Uniformly Distributed Load (cont.)

For distributed V and BM

Take section anywhere at distance, x from end

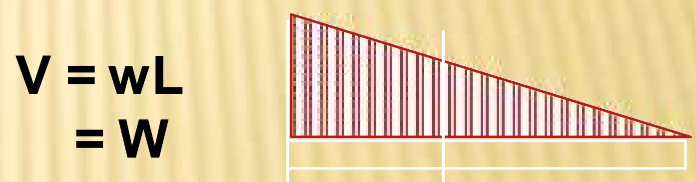
Add in forces, $V = w.x$ and moment $M = - wx.x/2$



Shear $V = wx$

when $x = L$ $V = W = wL$

when $x = 0$ $V = 0$



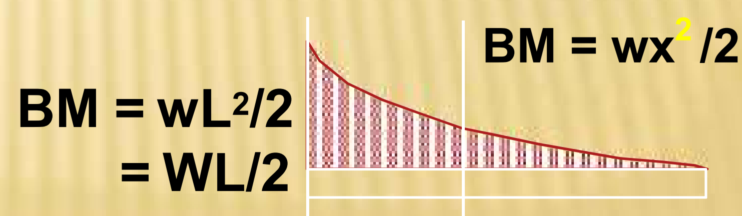
Shear Force Diagram

Bending Moment $BM = w.x^2/2$

when $x = L$ $BM = wL^2/2 = WL/2$

when $x = 0$ $BM = 0$

(parabolic)



Bending Moment Diagram

Fig. 6.22 shows a cantilever of length L fixed at A and carrying a gradually varying load from zero at the free end to w per unit length at the fixed end.

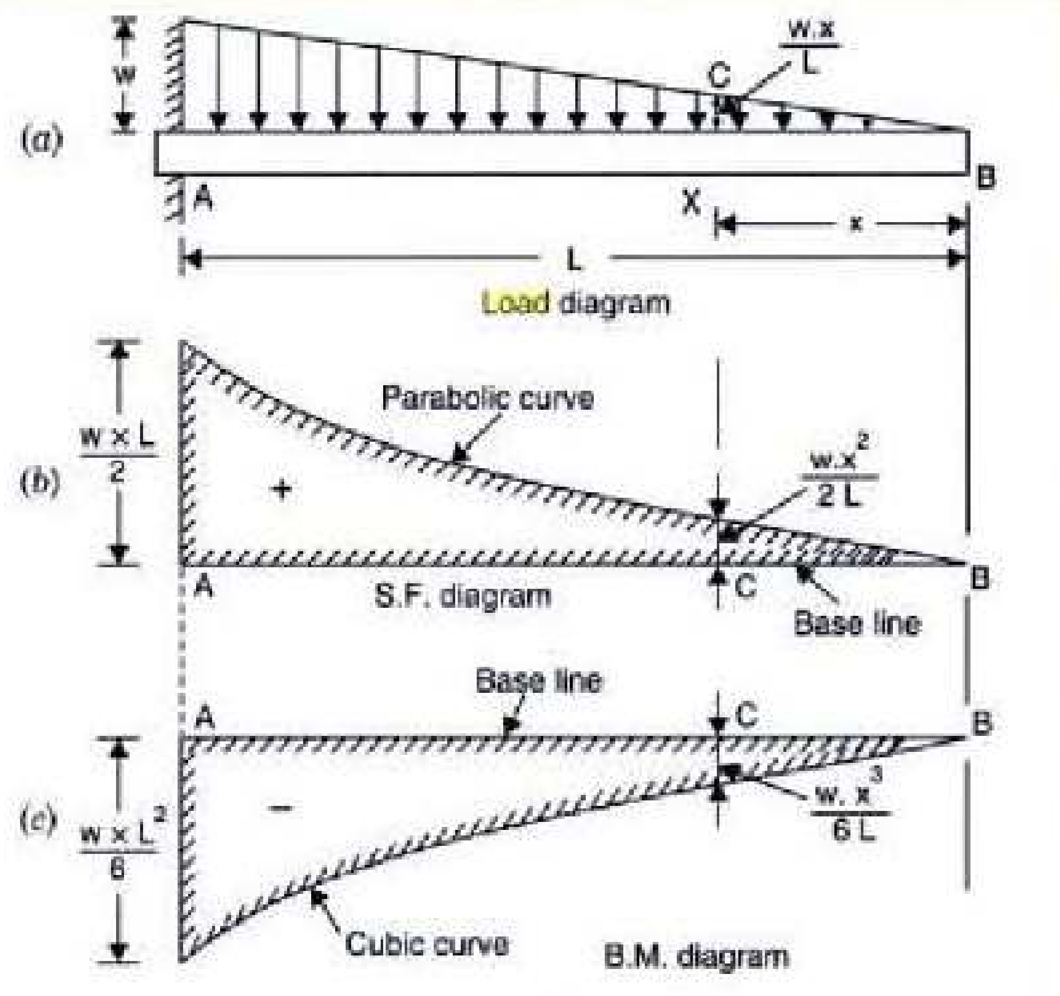


Fig. 6.22

Take a section X at a distance x from the free end B.

Let $\qquad F_x$ = Shear force at the section X, and

ce.

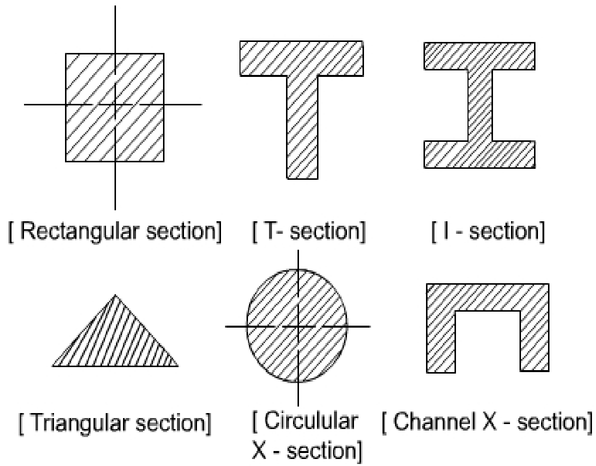
UNIT -3

Flexural and shear stresses
in beams

- **Members Subjected to Flexural Loads**
- **Introduction:**
- In many engineering structures members are required to resist forces that are applied laterally or transversely to their axes. These type of members are termed as beam.
- There are various ways to define the beams such as
- **Definition I:** A beam is a laterally loaded member, whose cross-sectional dimensions are small as compared to its length.
- **Definition II:** A beam is nothing simply a bar which is subjected to forces or couples that lie in a plane containing the longitudinal axis of the bar. The forces are understood to act perpendicular to the longitudinal axis of the bar.
- **Definition III:** A bar working under bending is generally termed as a beam.
- **Materials for Beam:**
- The beams may be made from several usable engineering materials such commonly among them are as follows:
- Metal
- Wood
- Concrete
- Plastic

Geometric forms of Beams:

- The Area of X-section of the beam may take several forms some of them have been shown below:



Loading restrictions:

Concept of pure bending:

- As we are aware of the fact internal reactions developed on any cross-section of a beam may consists of a resultant normal force, a resultant shear force and a resultant couple. In order to ensure that the bending effects alone are investigated, we shall put a constraint on the loading such that the resultant normal and the resultant shear forces are zero on any cross-section perpendicular to the longitudinal axis of the member,

That means F = 0

since or M = constant.

Thus, the zero shear force means that the bending moment is constant or the bending is same at every cross-section of the beam. Such a situation may be visualized or envisaged when the beam or some portion of the beam, as been loaded only by pure couples at its ends. It must be recalled that the couples are assumed to be loaded in the plane of symmetry.

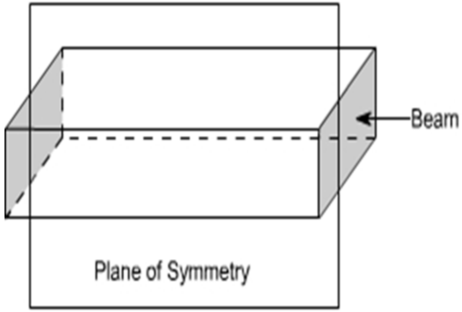


Fig (1)

Fig (2)

Bending Stresses in Beams or Derivation of Elastic Flexural formula :

- In order to compute the value of bending stresses developed in a loaded beam, let us consider the two cross-sections of a beam **HE** and **GF** , originally parallel as shown in fig 1(a).when the beam is to bend it is assumed that these sections remain parallel i.e. **H'E'** and **G'F'** , the final position of the sections, are still straight lines, they then subtend some angle
- Consider now fibre AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'

Therefore,

$$\text{strain in fibre AB} = \frac{\text{change in length}}{\text{orginal length}}$$

$$= \frac{A'B' - AB}{AB} \qquad \text{But } AB = CD \text{ and } CD = C'D'$$

refer to fig1(a) and fig1(b)

$$\therefore \text{strain} = \frac{A'B' - C'D'}{C'D'}$$

- Consider now fibre AB in the material, at a distance y from the N.A, when the beam bends this will stretch to A'B'
- Since CD and C'D' are on the neutral axis and it is assumed that the Stress on the neutral axis zero. Therefore, there won't be any strain on the neutral axis

$$= \frac{(R+y)\theta - R\theta}{R\theta} = \frac{R\theta + y\theta - R\theta}{R\theta} = \frac{y}{R}$$

However $\frac{\text{stress}}{\text{strain}} = E$ where E = Young's Modulus of elasticity

Therefore, equating the two strains as obtained from the two relations i.e,

$$\frac{\sigma}{E} = \frac{y}{R} \text{ or } \frac{\sigma}{y} = \frac{E}{R} \qquad \ldots\ldots\ldots\ldots(1)$$

$$\sigma = \frac{E}{R}\, y$$

if the shaded strip is of area 'dA'
then the force on the strip is

$$F = \sigma\, \delta A = \frac{E}{R}\, y\, \delta A$$

Moment about the neutral axis would be $= F.y = \frac{E}{R}\, y^2 \delta A$

The toatl moment for the whole cross-section is therefore equal to

$$M = \sum \frac{E}{R}\, y^2\, \delta A = \frac{E}{R} \sum y^2 \delta A$$

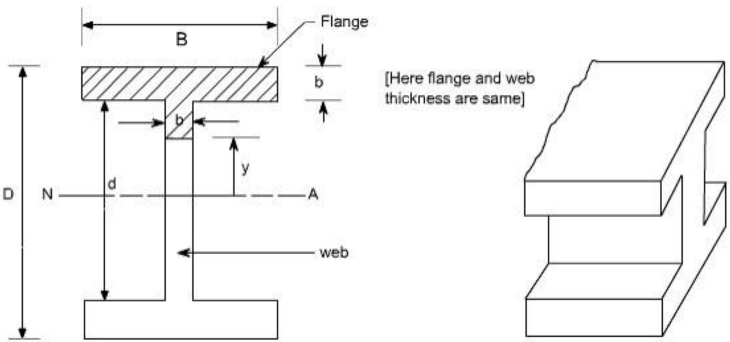
- Now the term is the property of the material and is called as a second moment of area of the cross-section and is denoted by a symbol I.
- Therefore $$M = \frac{E}{R} I \qquad \ldots\ldots\ldots(2)$$

combining equation 1 and 2 we get

$$\boxed{\frac{\sigma}{y} = \frac{M}{T} = \frac{E}{R}}$$

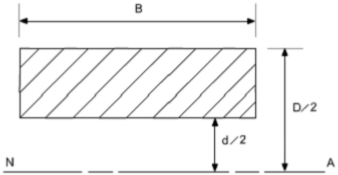
- **This equation is known as the Bending Theory Equation.** The above proof has involved the assumption of pure bending without any shear force being present. Therefore this termed as the pure bending equation. This equation gives distribution of stresses which are normal to cross-section i.e. in x-direction.

Consider an I - section of the dimension shown below.



The shear stress distribution for any arbitrary shape is given as $\tau = \dfrac{F A \bar{y}}{Z I}$

Let us evaluate the quantity $A\bar{y}$, the $A\bar{y}$ quantity for this case comprise the contribution due to flange area and web area



Flange area

Area of the flange $= B\left(\frac{D-d}{2}\right)$

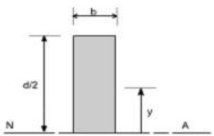
Distance of the centroid of the flange from the N.A

$$\bar{y} = \frac{1}{2}\left(\frac{D-d}{2}\right) + \frac{d}{2}$$

$$\bar{y} = \left(\frac{D+d}{4}\right)$$

Hence,

$$A\bar{y}\big|_{Flange} = B\left(\frac{D-d}{2}\right)\left(\frac{D-d}{4}\right)$$



Web Area

Area of the web

$$A = b\left(\frac{d}{2} - y\right)$$

Distance of the centroid from N.A

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} - y\right) + y$$

$$\bar{y} = \frac{1}{2}\left(\frac{d}{2} + y\right)$$

Therefore,

$$A\bar{y}\big|_{web} = b\left(\frac{d}{2} - y\right)\frac{1}{2}\left(\frac{d}{2} + y\right)$$

Hence,

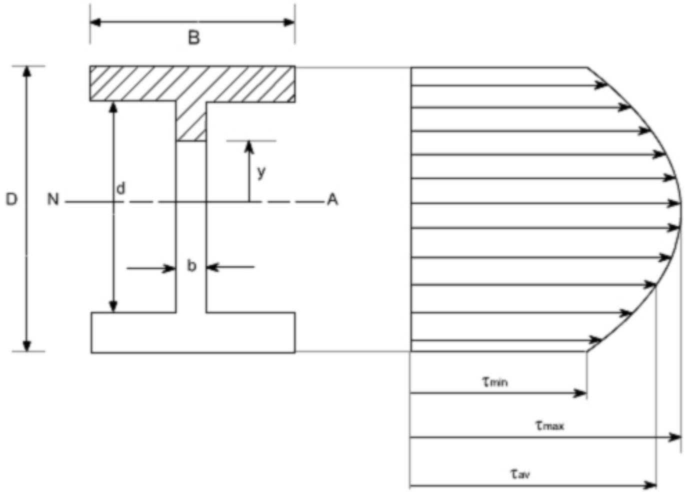
$$A\bar{y}\big|_{Total} = B\left(\frac{D-d}{2}\right)\left(\frac{D+d}{4}\right) + b\left(\frac{d}{2} - y\right)\left(\frac{d}{2} + y\right)\frac{1}{2}$$

Thus,

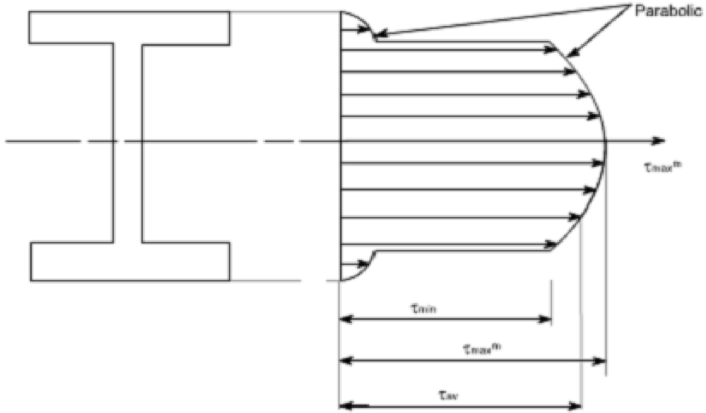
$$A\bar{y}\big|_{Total} = B\left(\frac{D^2-d^2}{8}\right) + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)$$

Therefore shear stress,

$$\tau = \frac{F}{bI}\left[\frac{B\left(D^2-d^2\right)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right)\right]$$



$$\tau_{max^m} = \frac{F}{8bI}\left[B\left(D^2 - d^2\right) + bd^2\right]$$



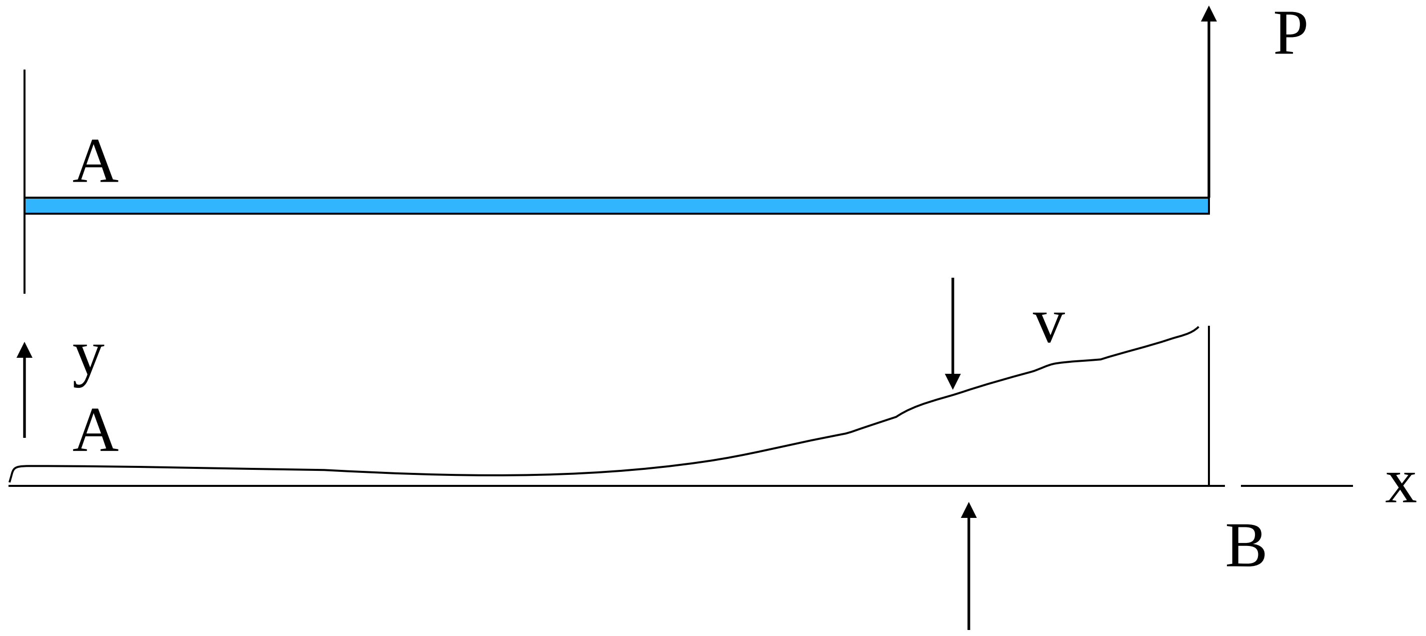
This distribution is known as the “top – hat” distribution. Clearly the web bears the most of the shear stress and bending theory we can say that the flange will bear most of the bending stress.

UNIT IV

DEFLECTION OF BEAMS

CHAPTER FIVE - DEFLECTION OF BEAMS

When a beam with a straight longitudinal axis is loaded by lateral forces, the axis is deformed into a curve, called the deflection curve of the beam. Deflection is the displacement in the y direction of any point on the axis of the beam. See Figure 5.1 below.



Deflection of Beams Contd.

* The calculation of deflections is an important part of structural analysis and design.
* Deflections are essential for example in the analysis of statically indeterminate structures and in dynamic analysis, as when investigating the vibration of aircraft or response of buildings to earthquakes.
* Deflections are sometimes calculated in order to verify that they are within tolerable limits.

5.1 RELATIONSHIP BETWEEN LOADING, SHEAR FORCE, BENDING MOMENT, SLOPE AND DEFLECTION.

Consider a beam AB which is initially horizontal when unloaded. If it deflects to a new position A ‘ B’ under load, the slope at any point C is:

$$i = dy/dx$$

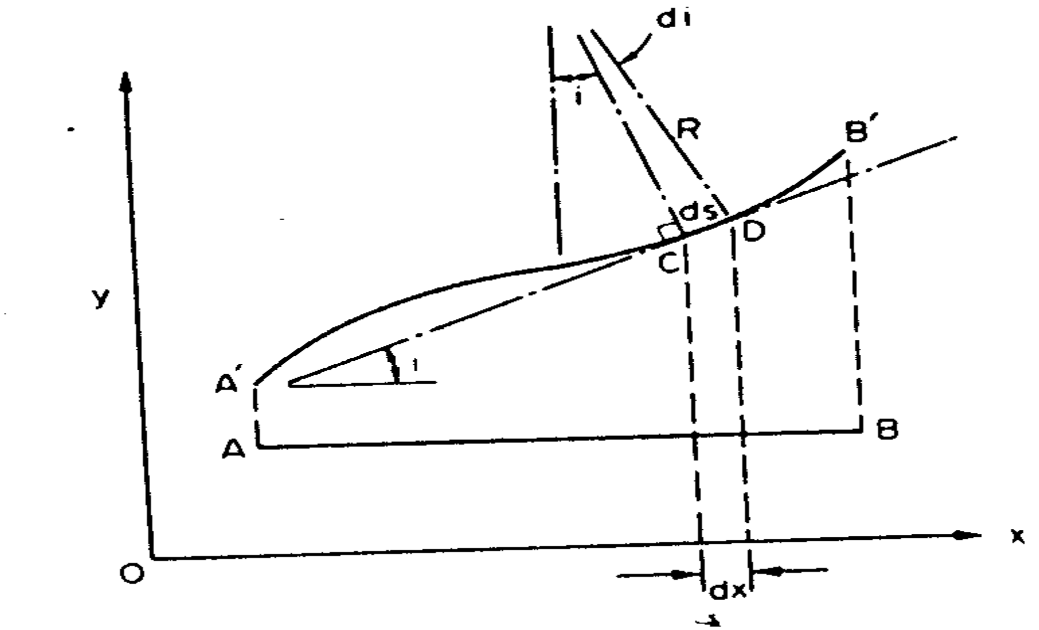


Fig. 5.2. Unloaded beam AB deflected to $A'B'$ under load.

Basic Differential Equation For Deflection

This is usually very small in practice, and for small curvatures:

$ds = dx = R\,di$ (Figure 5.2).

$$di/dx = 1/R$$

But $\quad i = dy/dx$

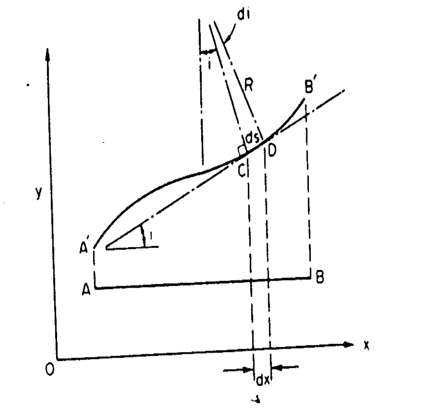


Fig. 5.2. Unloaded beam *AB* deflected to *A'B'* under load.

Therefore: $d^2y/dx^2 = 1/R$ ………………….

Now from simple bending theory: $M/I = E/R$

$$1/R = M/EI$$

Therefore substituting in equation (1):

$$M = E\,I\,d^2y/dx^2 \quad \text{……………………………………} (2)$$

This is the basic differential equation for the deflection of beams.

Recall that for a distributed load:

$dV/dx = -w$ (loading function) and $dM/dx = V$ (Shear force)

Differentiating once, $EI\, d^3y/dx^3 = dM/dx = V$

Differentiating further: $EI\, d^4y/dx^4 = dV/dx = -w$

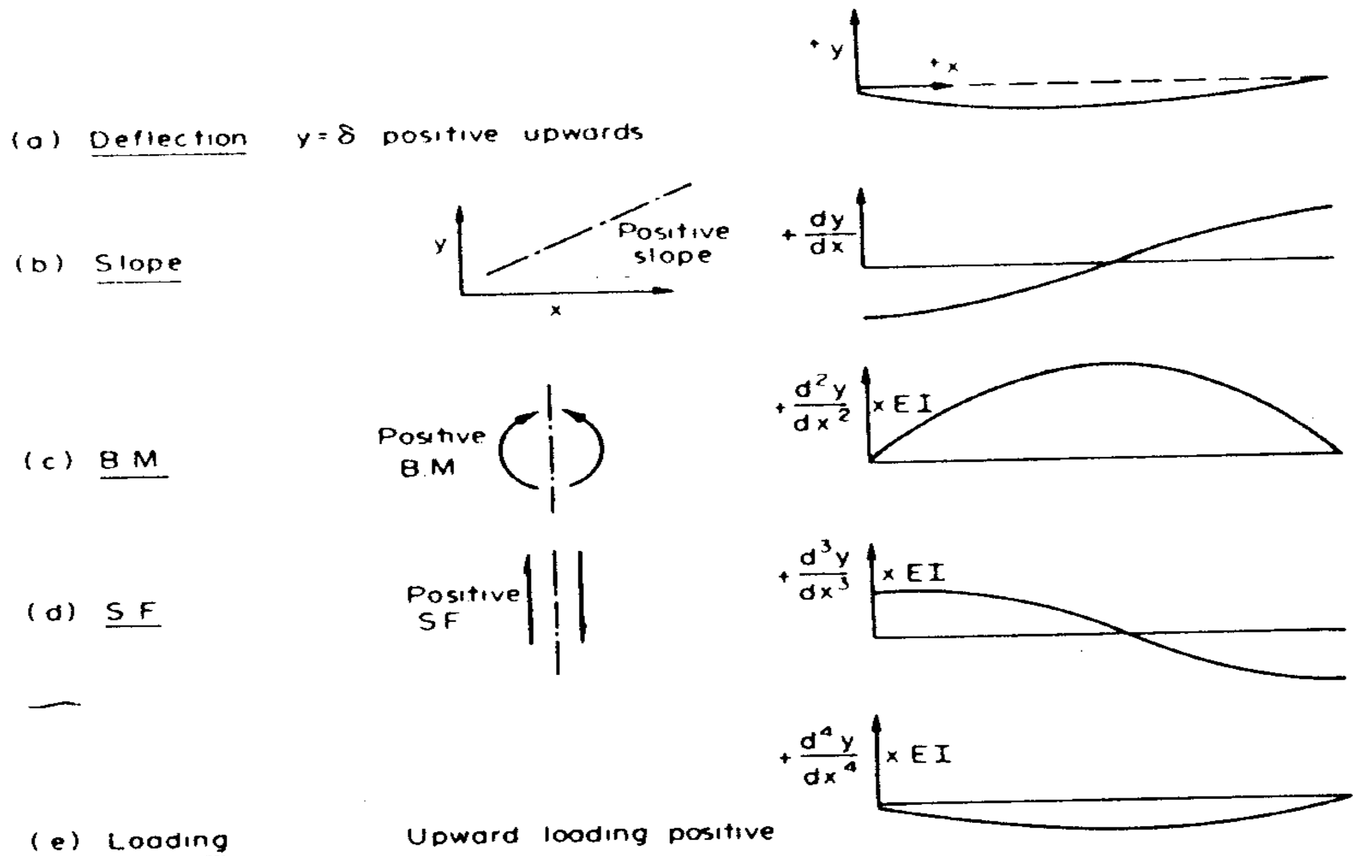


Fig. 5.4. Sign conventions for load, S.F., B.M., slope and deflection.

Direct Integration Method

5.2. Direct integration method

If the value of the B.M. at any point on a beam is known in terms of x, the distance along th beam, and provided that the equation applies along the complete beam, then integration o eqn. (5.4a) will yield slopes and deflections at any point,

i.e.
$$M = EI\frac{d^2y}{dx^2} \quad \text{and} \quad \frac{dy}{dx} = \int \frac{M}{EI}dx + A$$

or
$$y = \int\int\left(\frac{M}{EI}dx\right)dx + Ax + B$$

where A and B are constants of integration evaluated from known conditions of slope an deflection for particular values of x.

(a) *Cantilever with concentrated load at the end* (Fig. 5.5)

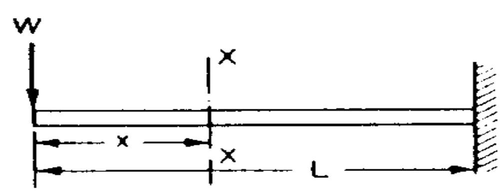


Fig. 5.5.

$$M_{xx} = EI\frac{d^2y}{dx^2} = -Wx$$

$$\therefore \qquad EI\frac{dy}{dx} = -\frac{Wx^2}{2} + A$$

assuming EI is constant.

$$EIy = -\frac{Wx^3}{6} + Ax + B$$

Now when $\qquad x = L, \quad \frac{dy}{dx} = 0 \qquad \therefore \quad A = \frac{WL^2}{2}$

and when $\qquad x = L, \; y = 0 \qquad \therefore \quad B = \frac{WL^3}{6} - \frac{WL^2}{2}L = -\frac{WL^3}{3}$

$$\therefore \qquad y = \frac{1}{EI}\left[-\frac{Wx^3}{6} + \frac{WL^2x}{2} - \frac{WL^3}{3}\right] \tag{5.5}$$

This gives the deflection at all values of x and produces a maximum value at the tip of the cantilever when $x = 0$,

i.e. $\qquad$ **Maximum deflection** $= y_{max} = -\dfrac{WL^3}{3EI}$ $\qquad$ (5.6)

The negative sign indicates that deflection is in the negative y direction, i.e. downwards.

Similarly $\qquad \dfrac{dy}{dx} = \dfrac{1}{EI}\left[-\dfrac{Wx^2}{2} + \dfrac{WL^2}{2}\right]$ $\qquad$ (5.7)

and produces a maximum value again when $x = 0$.

Maximum slope $= \left(\dfrac{dy}{dx}\right)_{max} = \dfrac{WL^2}{2EI}$ **(positive)** $\qquad$ (5.8) (*b*) *Cantilever with uniformly distributed load* (Fig. 5.6)

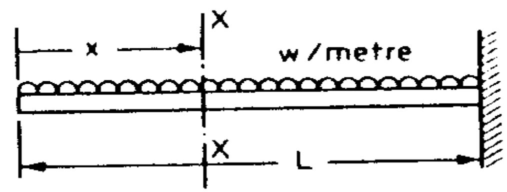


Fig. 5.6.

$$M_{xx} = EI\frac{d^2y}{dx^2} = -\frac{wx^2}{2}$$

$$EI\frac{dy}{dx} = -\frac{wx^3}{6} + A$$

$$EIy = -\frac{wx^4}{24} + Ax + B$$

Again, when $x = L, \quad \dfrac{dy}{dx} = 0 \quad \text{and} \quad A = \dfrac{wL^3}{6}$

$$x = L, \quad y = 0 \quad \text{and} \quad B = \frac{wL^4}{24} - \frac{wL^4}{6} = -\frac{wL^4}{8}$$

$$y = \frac{1}{EI}\left[-\frac{wx^4}{24} + \frac{wL^3x}{6} - \frac{wL^4}{8}\right] \tag{5.9}$$

At $x = 0$, $$y_{max} = -\frac{wL^4}{8EI} \quad \text{and} \quad \left(\frac{dy}{dx}\right)_{max} = \frac{wL^3}{6EI} \tag{5.10}$$

(c) *Simply-supported beam with uniformly distributed load* (Fig. 5.7)

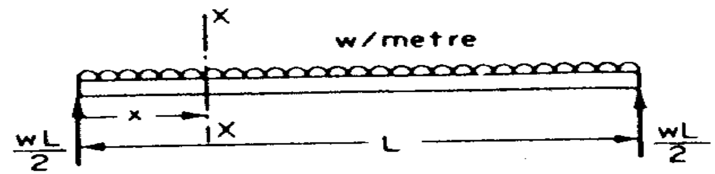


Fig. 5.7.

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{wLx}{2} - \frac{wx^2}{2}.$$

$$EI\frac{dy}{dx} = \frac{wLx^2}{4} - \frac{wx^3}{6} + A$$

$$EIy = \frac{wLx^3}{12} - \frac{wx^4}{24} + Ax + B$$

At $\quad x = 0, \quad y = 0 \qquad \therefore \quad B = 0$

At $\quad x = L, \quad y = 0 \qquad \therefore \quad 0 = \dfrac{wL^4}{12} - \dfrac{wL^4}{24} + AL$

$$\therefore \quad A = -\frac{wL^3}{24}$$

$$\therefore \quad y = \frac{1}{EI}\left[\frac{wLx^3}{12} - \frac{wx^4}{24} - \frac{wL^3x}{24}\right] \qquad (5.11)$$

In this case the maximum deflection will occur at the centre of the beam where $x = L/2$.

$$\therefore \quad y_{\max} = \frac{1}{EI}\left[\frac{wL}{12}\left(\frac{L^3}{8}\right) - \frac{w}{24}\left(\frac{L^4}{16}\right) - \frac{wL^3}{24}\left(\frac{L}{2}\right)\right]$$

$$= -\frac{5wL^4}{384EI} \qquad (5.12)$$

Similarly $\quad \left(\dfrac{dy}{dx}\right)_{\max} = \pm\dfrac{wL^3}{24EI}$ at the ends of the beam. $\qquad (5.13)$

(*d*) *Simply supported beam with central concentrated load* (Fig. 5.8)

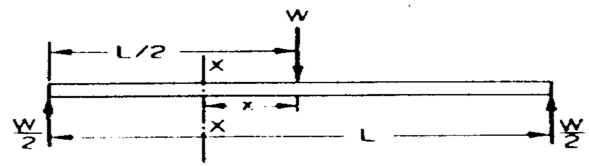


Fig. 5.8.

In order to obtain a single expression for B.M. which will apply across the complete beam in this case it is convenient to take the origin for x at the centre, then:

$$M_{xx} = EI\frac{d^2y}{dx^2} = \frac{W}{2}\left(\frac{L}{2} - x\right) = \frac{WL}{4} - \frac{Wx}{2}$$

$$EI\frac{dy}{dx} = \frac{WL}{4}x - \frac{Wx^2}{4} + A$$

$$EIy = \frac{WLx^2}{8} - \frac{Wx^3}{12} + Ax + B$$

At $\quad x = 0, \quad \frac{dy}{dx} = 0 \quad \therefore \quad A = 0$

$\quad x = \frac{L}{2}, \quad y = 0 \quad \therefore \quad 0 = \frac{WL^3}{32} - \frac{WL^3}{96} + B$

$$\therefore \quad B = -\frac{WL^3}{48}$$

$$\therefore \quad y = \frac{1}{EI}\left[\frac{WLx^2}{8} - \frac{Wx^3}{12} - \frac{WL^3}{48}\right] \tag{5.14}$$

$$\therefore \quad y_{max} = -\frac{WL^3}{48EI} \quad \text{at the centre} \tag{5.15}$$

and

$$\left(\frac{dy}{dx}\right)_{max} = \pm\frac{WL^2}{16EI} \quad \text{at the ends} \tag{5.16}$$

Direct Integration Method Contd.

* In Some cases, it is not convenient to commence the integration procedure with the bending moment equation since this may be difficult to obtain. In such cases, it is often more convenient to commence with the equation for the loading at the general point XX of the beam. A typical example follows:

(c) *Cantilever subjected to non-uniform distributed load* (Fig. 5.9)

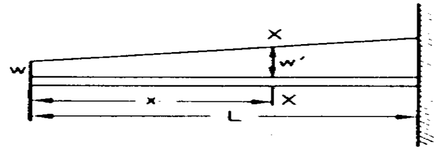
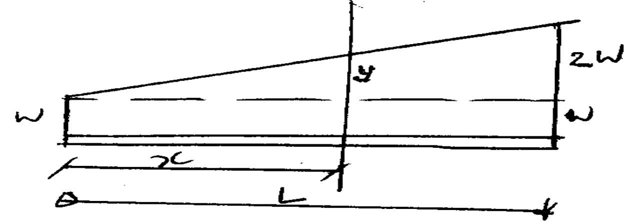


Fig. 5.9.



Load $= W + 2W\frac{x}{L}$

$= W\left[1+\frac{2x}{L}\right]$

The loading at section XX is

$$w' = EI\frac{d^4y}{dx^4} = -\left[w + (3w - w)\frac{x}{L}\right] = -w\left(1 + \frac{2x}{L}\right)$$

Integrating,

S.F: $$EI\frac{d^3y}{dx^3} = -w\left(x + \frac{x^2}{L}\right) + A \qquad (1)$$

BM: $$EI\frac{d^2y}{dx^2} = -w\left(\frac{x^2}{2} + \frac{x^3}{3L}\right) + Ax + B \qquad (2)$$

Slope: $$EI\frac{dy}{dx} = -w\left(\frac{x^3}{6} + \frac{x^4}{12L}\right) + \frac{Ax^2}{2} + Bx + C \qquad (3)$$

defln: $$EIy = -w\left(\frac{x^4}{24} + \frac{x^5}{60L}\right) + \frac{Ax^3}{6} + \frac{Bx^2}{2} + Cx + D \qquad (4)$$

Thus, before the slope or deflection can be evaluated, four constants have to be determined; therefore four conditions are required. They are:

At $x = 0$, S.F. is zero
from (1) $A = 0$
At $x = 0$, B.M. is zero
from (2) $B = 0$
At $x = L$, slope $dy/dx = 0$ (slope normally assumed zero at a built-in support)

from (3) $$0 = -w\left(\frac{L^3}{6} + \frac{L^3}{12}\right) + C$$

$$C = \frac{wL^3}{4}$$

At $x = L$, $y = 0$

from (4) $$0 = -w\left(\frac{L^4}{24} + \frac{L^4}{60}\right) + \frac{wL^4}{4} + D$$

$$D = -\frac{23wL^4}{120}$$

$$\therefore \quad EIy = -\frac{wx^4}{24} - \frac{wx^5}{60L} + \frac{wL^3x}{4} - \frac{23wL^4}{120}$$

Then, for example, the deflection at the tip of the cantilever, where $x = 0$, is

$$y = -\frac{23wL^4}{120EI}$$

Macaulay's Method

* The Macaulay's method involves the general method of obtaining slopes and deflections (i.e. integrating the equation for M) will still apply provided that the term, W (x – a) is integrated with respect to (x – a) and not x.

Example of Using Macaulay's Method for Concentrated loads

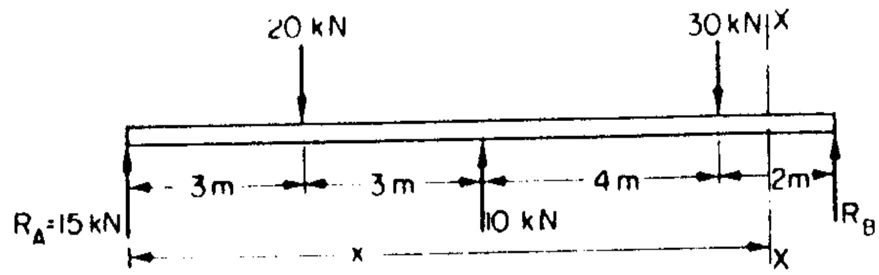


Fig. 5.11.

As an illustration of the procedure consider the beam loaded as shown in Fig. 5.11 for which the central deflection is required. Using the Macaulay method the equation for the B.M. at any general section XX is then given by

$$\text{B.M.}_{XX} = 15x - 20[(x-3)] + 10[(x-6)] - 30[(x-10)]$$

Care is then necessary to ensure that the terms inside the square brackets (Macaulay terms) are treated in the special way noted on the previous page.

Here it must be emphasised that all loads in the right-hand side of the equation are in units of kN (i.e. newtons $\times 10^3$). *In subsequent working, therefore, it is convenient to carry through this factor as a denominator on the left-hand side in order that the expressions are dimensionally correct.*

Example Contd.

Integrating,

$$\frac{EI}{10^3}\frac{dy}{dx} = 15\frac{x^2}{2} - 20\left[\frac{(x-3)^2}{2}\right] + 10\left[\frac{(x-6)^2}{2}\right] - 30\left[\frac{(x-10)^2}{2}\right] + A$$

and $$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] + Ax + B$$

where A and B are two constants of integration.

Now when $x = 0$, $y = 0$ $\therefore$ $B = 0$

and when $x = 12$, $y = 0$

$$\therefore \quad 0 = \frac{15 \times 12^3}{6} - 20\left[\frac{9^3}{6}\right] + 10\left[\frac{6^3}{6}\right] - 30\left[\frac{2^3}{6}\right] + 12A$$

$$= 4320 - 2430 + 360 - 40 + 12A$$

$$\therefore \quad 12A = -4680 + 2470 = -2210$$

$$\therefore \quad A = -184.2$$

The deflection at any point is given by

$$\frac{EI}{10^3}y = 15\frac{x^3}{6} - 20\left[\frac{(x-3)^3}{6}\right] + 10\left[\frac{(x-6)^3}{6}\right] - 30\left[\frac{(x-10)^3}{6}\right] - 184.2x$$

The deflection at mid-span is thus found by substituting $x = 6$ in the above equation, bearing in mind that the dimensions of the equation are kN m^3.

N.B. – Two of the Macaulay terms then vanish since one becomes zero and the other negative and therefore neglected.

$$\therefore \quad \text{central deflection} = \frac{10^3}{EI}\left[\frac{15 \times 6^3}{6} - \frac{20 \times 3^3}{6} - 184.2 \times 6\right]$$

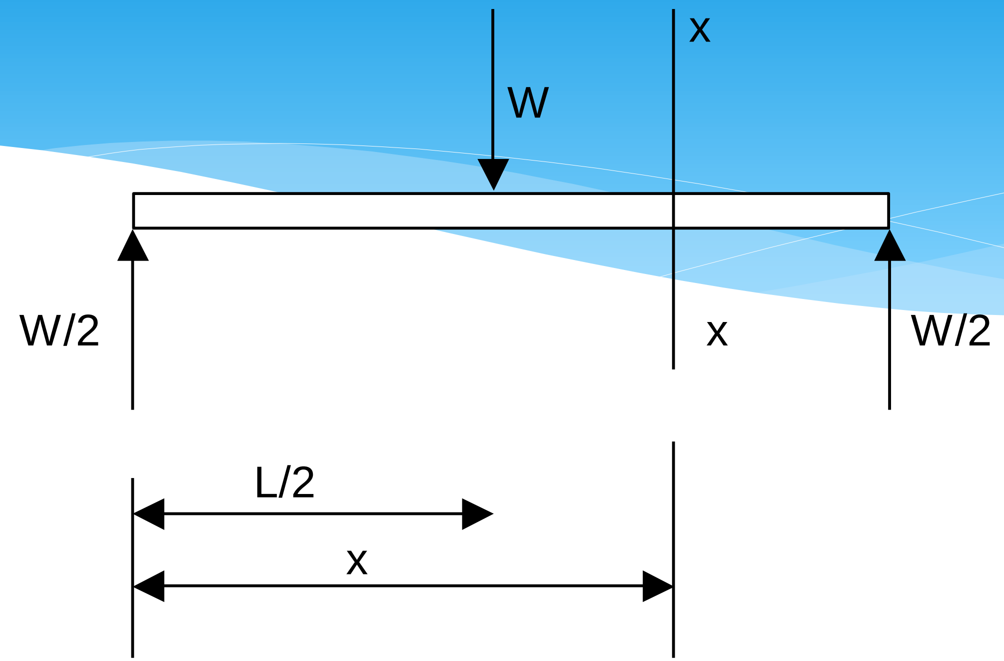
$$= -\frac{655.2 \times 10^3}{EI}$$

55.4

With typical values of $E = 208$ GN/m^2 and $I = 82 \times 10^{-6}$ m^4

central deflection $= 38.4 \times 10^{-3}$ m $=$ **38.4 mm**

Using Macaulay's Method to Solve the Situation of Simply Supported Beam with Concentrated Load



$$M_x = W/2\,x - W\,[x - L/2]$$

$$E\,I\,d^2y/dx^2 = W/2\,x - W\,[x - L/2]$$

$$E\,I\,dy/dx = W/4\,x^2 - \frac{W\,[x - L/2]^2}{2} + A$$

$$E\,I\,y = \frac{W\,x^3}{12} - \frac{W\,[x - L/2]^3}{6} + A\,x + B$$

Boundary Conditions

At $x = 0$, $y = 0$ i.e. $B = 0$

At $x = 0$, $y = 0$

$$\text{i.e.} \quad 0 = \frac{WL^3}{12} - \frac{W[L - L/2]^3}{6} + AL$$

$$0 = = \frac{WL^3}{12} - \frac{W L^3}{48} + AL$$

$$A = \frac{WL^2}{48} - \frac{WL^2}{12} = \frac{-WL^2}{16}$$

$$E I y = \frac{W x^3}{12} - \frac{W[x - L/2]^3}{6} - \frac{WL^2 x}{16}$$

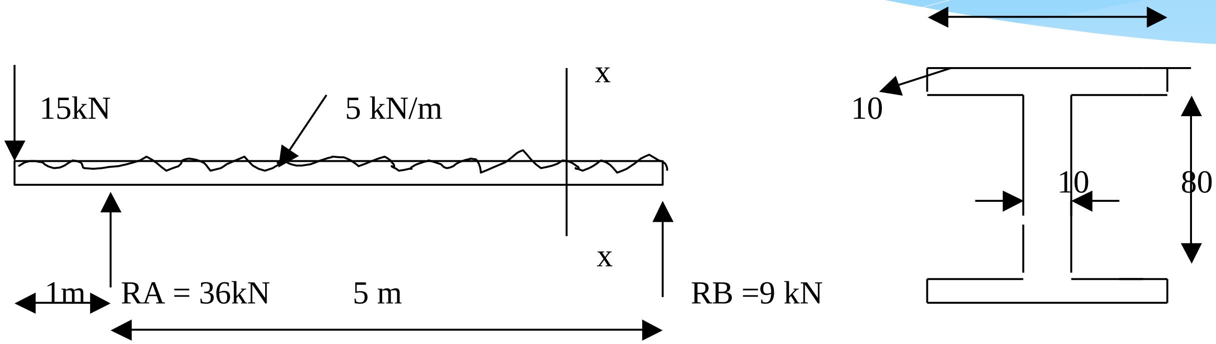
Ymax occurs at $x = L/2$

$$\text{i.e.} \quad E I y = \frac{W}{12} \frac{[L^3)}{8} \frac{- WL^2}{16} \frac{[L]}{2}$$

$$Y \max = \frac{- W L^3}{48 EI}$$

Macaulay’s Method for u.d.l.s

Example: Determine the deflection of the beam at x = 3 (middle of Beam) if E = 210 kN mm^{-2}. The Cross-Section is given as below.



Taking Moment about B: $15 \times 6 - 5\,RA + (5 \times 6) \times 3 = 0$

$RA = 36$ kN

$RB = 15 + (5 \times 6) - 36 = 9$ kN

$EI\, d^2y/dx^2 = -15\,x + 36\,[x-1] - 5\,x^2/2$

$EI\, dy/dx = -15/2\,x^2 + 36/2\,[x-1]^2 - 5/6\,x^3 + A$

$E\,I\,y = -15/6\,x^3 + 36/6\,[x-1]^3 - 5/24\,x^4 + A\,x + B$

$EI\,y = -2.5\,x^3 + 6\,[x-1]^3 - 0.2083\,x^4 + A\,x + B$

Boundary Conditions

At $x = 1, \; y = 0$

i.e. $0 = -2.5 - 0.21 + A + B$

i.e. $A + B = 2.5 + 0.2083 = 2.71$ ……….. (1)

Also: At $x = 6, \; y = 0$

i.e. $0 = -540 + 750 - 272.16 + 6A + B$

i.e . $6A + B = 60$ ------------- (2)

From Equations (1) and (2), $A = 11.45$ and $B = -8.75$

i.e. $E I y = -2.5 x^3 + 6 [x - 1]^3 - 0.2083 x^4 + 11.46 x - 8.75$

At $x = 3$, $EI y = (-2.5 \times 27) + 48 - 16.87 + 34.38 - 8.75$

$= -10.74$

Solution Concluded

Moment of Inertia of given section about the neutral axis

$$= 2\left[\frac{50\ x\ 10^3}{12} + 500\ \ x\ \ 45^2\right] + \frac{10\ x\ 80^3}{12} = 2.46x10^{-6}\ \ m^4$$

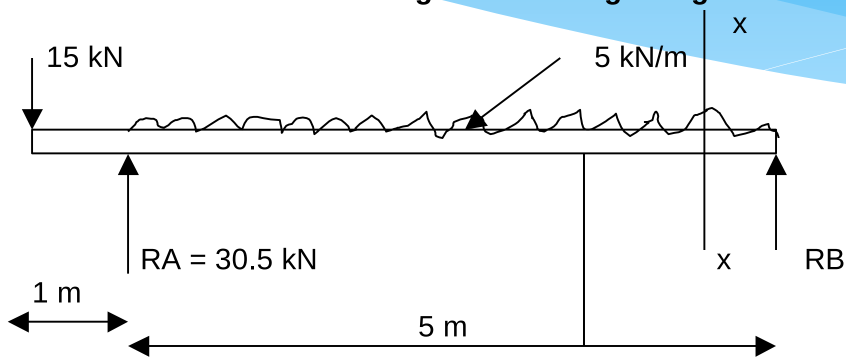
$$y = \frac{-10.74\quad kNm^3}{E\quad I} = \frac{-10.74\quad kNm^3}{210\ x\ 10^6\,kN/m^2\quad x\quad 2.46\ x10^{-6}\quad m^4}$$

= 0.02079 m = **<u>20.79 mm</u>**

Two Special Cases

TWO SPECIAL CASES

CASE 1: Uniform Load Not Starting from the Beginning



Taking Moment about B: 15 x 6 - 5 RA + 5 x 5 x 2.5 = 0

RA = 30.5 kN

RB = 15 + 25 - 30.5 = 9.5 kN

$$EI\, d^2y/dx^2 = -15x + 30.5\,[x-1] - 5\,[x-1] \cdot \frac{[x-1]}{2}$$

$$EI\, d^2y/dx^2 = -15x + 30.5\,[x-1] - \frac{5\,[x-1]^2}{2}$$

$$EI\, dy/dx = -15/2\; x^2 + 30.5/2\;[x-1]^2 - \frac{5\,[x-1]^3}{6} + A$$

$$EI\, y = -15/6\; x^3 + 30.5/6\;[x-1]^3 - \frac{5\,[x-1]^4}{24} + A x + B$$

Solution of Case 1 Concluded

Boundary Conditions

At $x = 1$, $y = 0$ i.e. $0 = -2.5 + A + B$

i.e. $A + B = 2.5$ and $6A + 6B = 15$ ………… (1)

At $x = 6$, $y = 0$

i.e. $0 = -540 + 635.42 - 130.21 + 6A + B$

i.e. $6A + B = 34.79$ ……….. (2)

From Equations (1) and (2): $A = 6.46$ and $B = -3.96$

i.e. $EI\,y = -2.5\, x^3 + 5.083\, [x - 1]^3 - 0.2083\, [x - 1]^4 + 6.46\,x - 3.96$

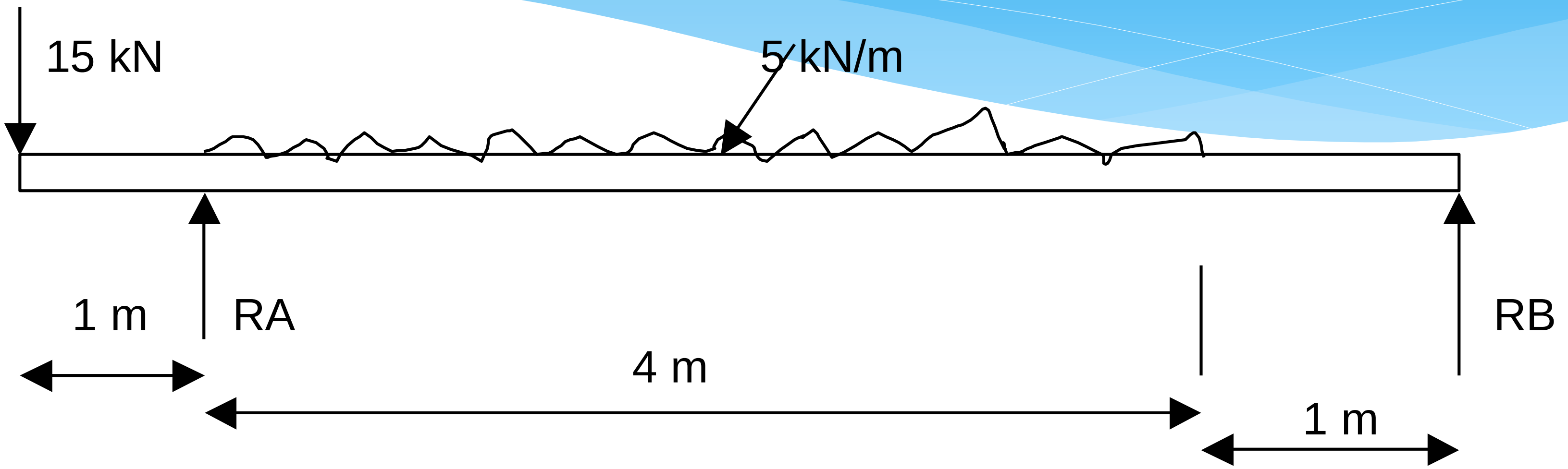
At $x = 3$ (mid-span)

$E\,I\,y = -67.5 + 40.66 - 3.33 + 19.38 - 3.96 = -14.75 \text{ kN m}^3$

$$y = \frac{-14.75 \quad kNm^3}{E \quad I} = \frac{-14.75 \quad kNm^3}{210\,x\,10^6\,kN/m^2 \quad x \quad 2.46 \quad x \quad 10^{-6} \quad m^4}$$

= **<u>28.55 mm</u>**

Case 2: Uniform Load Not reaching End of Beam



Taking Moment about B: 15 x 6 - 5 RA + 5 x 4 x 3 = 0

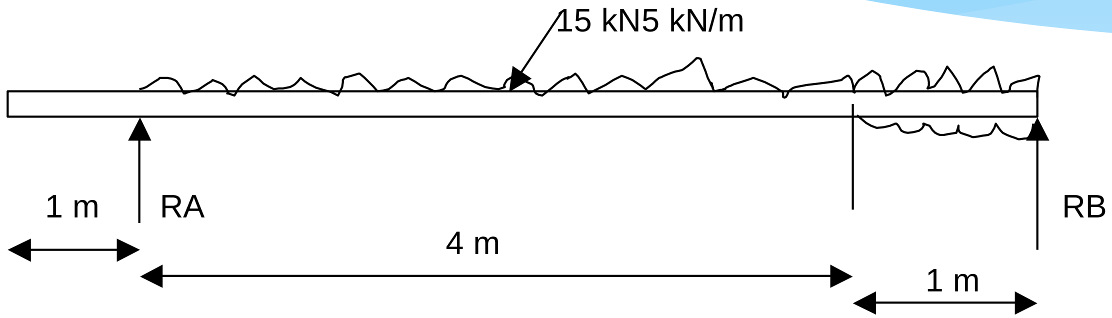
RA = 30 kN

RB = 15 + (5 x 4) - 30 = 5 kN

Since the 5 kN/m load did not reach the end, [x - 1] does not represent the actual loading.

Solution to Case 2 Contd.

Loading is equivalent to:



$$EI\, d^2y/dx^2 = -15x + 30[x-1] - 5[x-1]\cdot\frac{[x-1]}{2} + \frac{5[x-5][x-5]}{2}$$

$$EI\, d^2y/dx^2 = -15x + 30[x-1] - 5\frac{[x-1]^2}{2} + \frac{5[x-5]^2}{2}$$

$$EI\, dy/dx = -15/2\, x^2 + 15[x-1]^2 - 5\frac{[x-1]^3}{6} + \frac{5[x-5]^3}{6} + A$$

$$EI\, dy/dx = -15/6\, x^3 + 5[x-1]^3 - 5\frac{[x-1]^4}{24} + \frac{5[x-5]^4}{24} + A$$

Continue as usual to obtain y at 3 m.

Mohr's Area-Moment Method

* The Mohr area-moment procedure can be summarised as:
* If A and B are two points on the deflection curve of a beam, EI is constant and B is a point of zero slope, then the Mohr's theorems state that:
* (1) Slope at A = 1/EI x area of B.M. diagram between A and B
* (2) Deflection at A relative to B = 1/EI x first moment of area of B.M diagram between A and B about A.

(a) *Cantilever with concentrated load at the end*

In this case B is a point of zero slope and the simplified form of the Mohr theorems stated above can be applied.

$$\text{Slope at } A = \frac{1}{EI}\,[\text{area of B.M. diagram between } A \text{ and } B \text{ (Fig. 5.20)}]$$

$$= \frac{1}{EI}\left[\frac{L}{2}WL\right] = \frac{WL^2}{2EI}$$

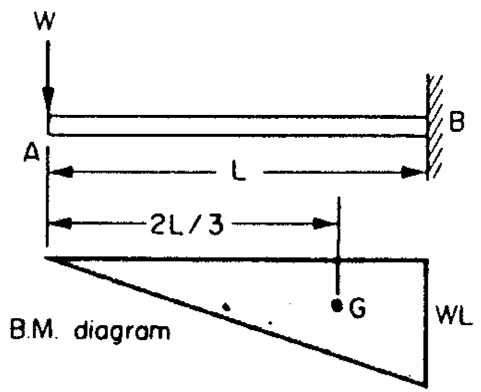


Fig. 5.20.

Deflection at A (relative to B)

$$= \frac{1}{EI}\,[\text{first moment of area of B.M. diagram between } A \text{ and } B \text{ about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{L}{2}WL\right)\frac{2L}{3}\right] = \frac{WL^3}{3EI}$$

(*b*) *Cantilever with u.d.l.*

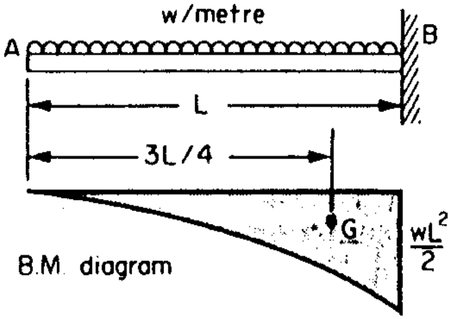


Fig. 5.21.

Parabolic Spandrel
(pp. 211, Text bk).

Again B is a point of zero slope.

$$\therefore \quad \text{slope at } A = \frac{1}{EI}[\text{area of B.M. diagram (Fig. 5.21)}]$$

$$= \frac{1}{EI}\left[\frac{1}{3}L\frac{wL^2}{2}\right]$$

$$= \frac{wL^3}{6EI}$$

$$\text{Deflection at } A = \frac{1}{EI}[\text{moment of B.M. diagram about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{1}{3}L\frac{wL^2}{2}\right)\frac{3L}{4}\right] = \frac{wL^4}{8EI}$$

Simply supported beam with u.d.l.

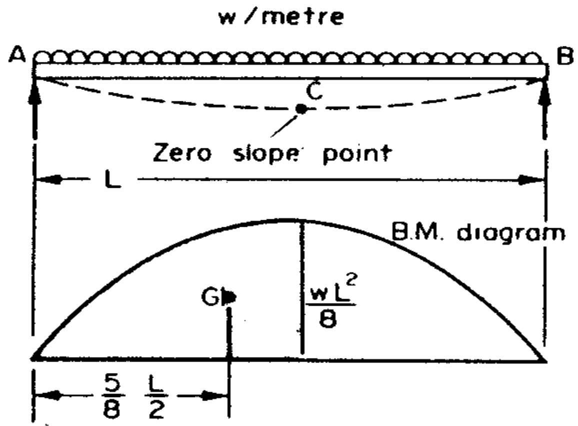


Fig. 5.22.

Semi-Parabolic area (pp 21, Text-bk)

Here the point of zero slope is at the centre of the beam C. Working relative to C,

$$\text{slope at } A = \frac{1}{EI}\,[\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.22)}]$$

$$= \frac{1}{EI}\left[\frac{2}{3}\frac{wL^2}{8}\frac{L}{2}\right] = \frac{wL^3}{24EI}$$

Deflection of A relative to C (= central deflection relative to A)

$$= \frac{1}{EI}\,[\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

$$= \frac{1}{EI}\left[\left(\frac{2}{3}\frac{wL^2}{8}\frac{L}{2}\right)\left(\frac{5L}{16}\right)\right] = \frac{5wL^4}{384EI}$$

(d) *Simply supported beam with central concentrated load*

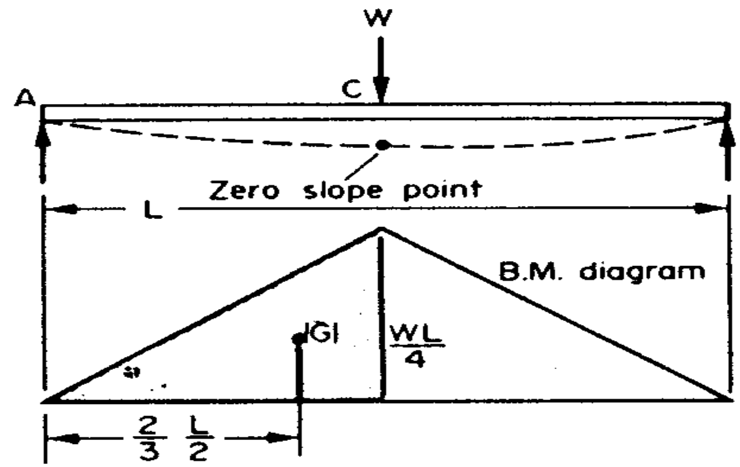


Fig. 5.23.

Again working relative to the zero slope point at the centre C,

$$\text{slope at } A = \frac{1}{EI}[\text{area of B.M. diagram between } A \text{ and } C \text{ (Fig. 5.23)}]$$

$$= \frac{1}{EI}\left[\frac{1}{2}\frac{L}{2}\frac{WL}{4}\right] = \frac{WL^2}{16EI}$$

Deflection of A relative to C (= central deflection of C)

$$= \frac{1}{EI}[\text{moment of B.M. diagram between } A \text{ and } C \text{ about } A]$$

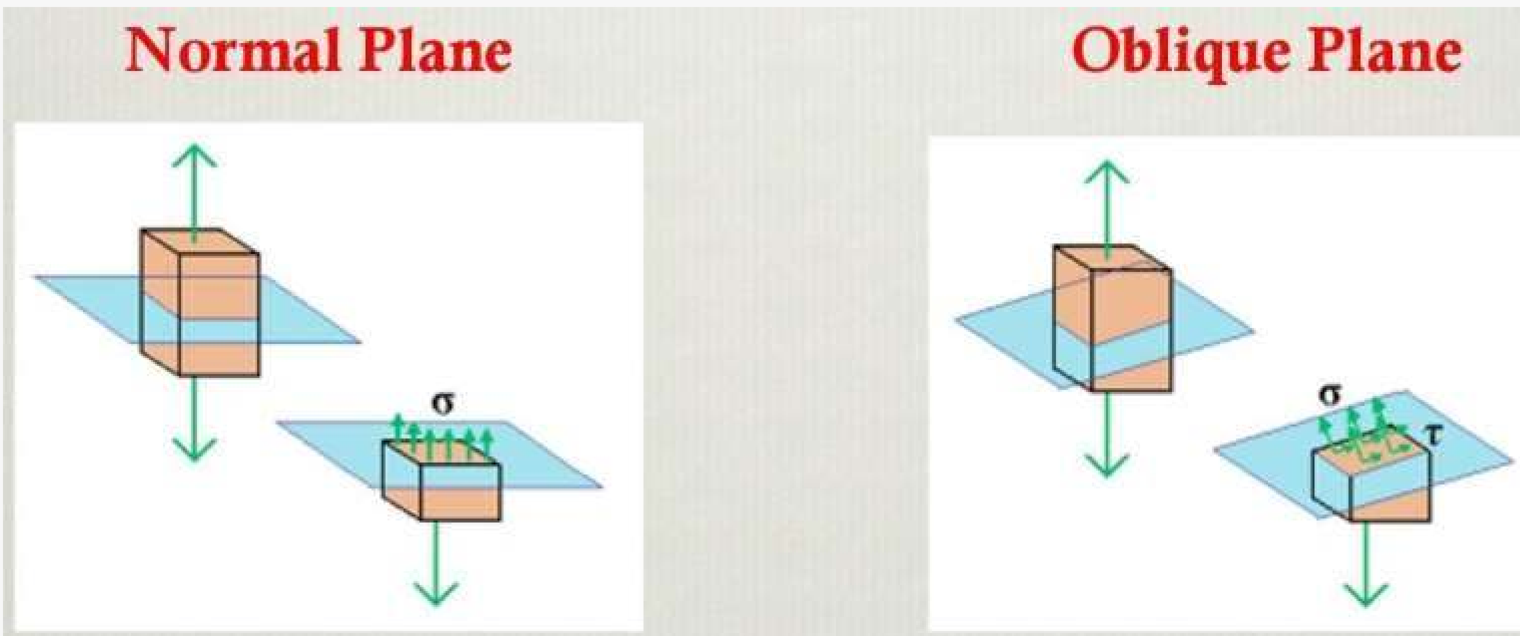
$$= \frac{1}{EI}\left[\left(\frac{1}{2}\frac{L}{2}\frac{WL}{4}\right)\left(\frac{2}{3}\frac{L}{2}\right)\right] = \frac{WL^3}{48EI}$$

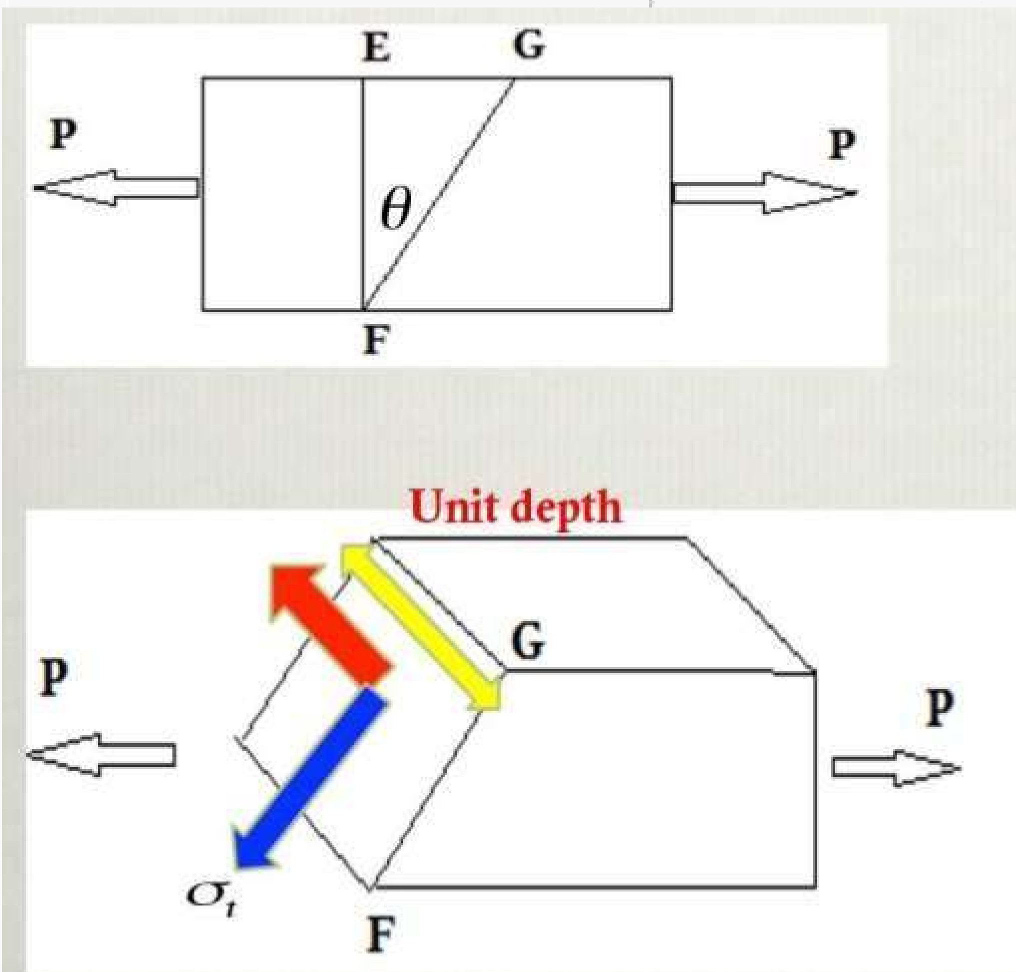
Principal stresses and strains

- What are principal stresses.
- Planes that have no shear stress are called as principal planes.
- Principal planes carry only normal stresses

Stresses in oblique plane

- In real life stresses does not act in normal direction but rather in inclined planes.





$\sigma = \frac{P}{A}$

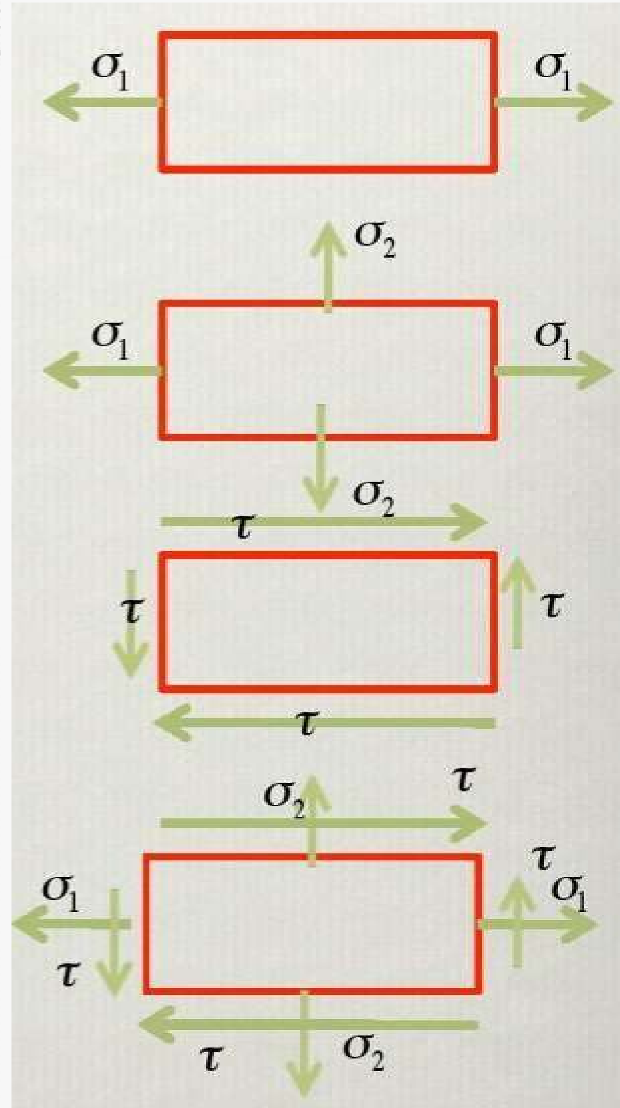
P = Axial forces

A = cross sectional area

$\sigma_n = \sigma\cos^2\theta$

$\sigma_t = \frac{\sigma}{2}\sin 2\theta$

- Member subjected to direct stress in one plane
- Member subjected to direct stress in two mutually perpendicular plane.
- Member subjected to simple shear stress.
- Member subjected to direct stress in two mutually perpendicular directions + simple shear stress.



- Member subjected to direct stress in two mutually

  perpendicular directions + simple shear stress

$$\sigma n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$$

- Member subjected to direct stress in two mutually

  perpendicular directions + simple shear stress

- POSITION OF PRINCIPAL PLANES
- Shear stress should be zero

$$\sigma_t = \frac{\sigma_{1-\sigma 2}}{2} \sin 2\theta - \tau \cos 2\theta = 0$$

$$\tan 2\theta = 2T/(\sigma_1 - \sigma_2)$$

- Member subjected to direct stress in two mutually perpendicular directions + simple shear stress .

Major principal Stress= $\frac{\sigma_1+\sigma_2}{2} \pm \frac{\sigma_1-\sigma_2}{2} \pm T$

Minor principal Stress = $\frac{\sigma_1+\sigma_2}{2} \pm \frac{\sigma_1-\sigma_2}{2} \pm T$

- Member subjected to direct stress in two mutually perpendicular directions + simple shear stress

- MAX SHEAR STRESS

$$\frac{d}{d\theta}(\sigma_t) = 0$$

$$\frac{d}{d\theta}[\tan 2\theta \sin 2\theta - \tau \cos 2\theta] = 0$$

$$\tan 2\theta = \frac{\sigma_1 - \sigma_2}{2T}$$

- Member subjected to direct stress in two mutually perpendicular directions + simple shear stress

- MAX SHEAR STRESS

$$\sigma_t = \frac{\sigma_1 - \sigma_2}{2} \sin 2\theta - \tau \cos 2\theta$$

$$\tan 2\theta = \frac{\sigma_1 - \sigma_2}{2T}$$

$$\sigma_{t(max)} = \frac{1}{2} \sqrt{(\sigma_1 - \sigma_2)^2 + 4T^2}$$

- Member subjected to direct stress in one plane
- Member subjected to direct stress in two mutually

  perpendicular plane
- Member subjected to simple shear stress.
- Member subjected to direct stress in two mutually

  perpendicular directions + simple shear stress

❖ Member subjected to direct stress in one plane

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$$

Stress in one direction and no shear stress $\sigma_2 = 0, \tau = 0$

$$\sigma_n = \frac{\sigma_1}{2} + \frac{\sigma_1}{2}\cos 2\theta = \sigma_1 \cos^2\theta$$

$$\sigma_t = \frac{\sigma_1}{2} \sin 2\theta$$

- Member subjected to direct stress in two mutually perpendicular plane

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$$

Stress in two direction and no shear stress $\tau=0$

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta$$

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta$$

❖ Member subjected to simple shear stress.

$$\sigma_n = \frac{\sigma_1+\sigma_2}{2} + \frac{\sigma_1-\sigma_2}{2}\cos 2\theta + \tau \sin 2\theta$$

$$\sigma_t = \frac{\sigma_1-\sigma_2}{2}\sin 2\theta - \tau \cos 2\theta$$

No stress in axial direction but only shear stress σ1=σ2 =0

$$\sigma_n = \tau \sin 2\theta$$

$$\sigma_t = -\tau \cos 2\theta$$

THEORIES OF FAILURE

- ✓ In case of material subjected to simple state of stress (tension or compression), failure occurs when the stress in the material reaches the elastic limit stress.
- ✓ In case of material subjected to complex stresses, the stage of failure is determined either to practically or theoretically.
- ✓ Non-applicability of any one theory to all states of stresses and to all materials has resulted in propagation of different theories relating the complex stresses to elastic limit in simple tension or compression.
- ✓ Since the complex stress system can be simplified into three principal stresses, the problem reduced to linking the three principal stresses to the stresses at elastic limit in case of simple stresses.

- The common most theories are

1. Maximum principal stress theory
2. Maximum Principal strain theory
3. Maximum shear stress theory
4. Maximum strain energy theory
5. Maximum shear strain energy theory

1. Maximum Principal stress theory
or
Maximum normal stress theory

- This theory was proposed by Rankine.
- It states that failure will occur when the maximum principal stress (σ_1) in the complex system reaches the value of maximum stress (σ_{yt}) at the elastic limit simple tension or the minimum principal stress (i.e. maximum principal compression stress) reaches the elastic limit (σ_{yc}) in simple compression.

$\sigma_1 = \sigma_{yt}$ in simple tension

$|\sigma_3| = \sigma_{yc}$ in simple compression

- For the design, the maximum principal stress should not exceed the working stress σ for the material. $\sigma_1 \leq \sigma$,
- Working stress, $\sigma = \dfrac{\sigma_y}{F}$

F : Factor of safety

- This theory is valid for brittle metals such as cast iron.
- Maximum principal stress theory is valid for thin walled tubes.

The maximum principal stress theory is contradicted in the following cases.

i. Failure in simple tension is caused by sliding at 45^0 with the axis of the specimen, there by failure occurred due to maximum shear stress and not due to direct tensile stress.

ii. The material which is weak in simple compression can sustain large hydrostatic pressure in excess of the elastic limit in simple compression.

Maximum principal strain theory

- This theory was proposed by Saint Venant.
- It states that the failure of a material occurs when the major principal tensile strain reaches the strain at the elastic limit in simple tension or when the minor principal strain (i.e maximum principal compressive strain) reaches the strain at elastic limit in simple compression.
- This theory is more appropriate for ductile materials, brittle materials and materials under hydrostatic pressure.
- It does not fit well with the experimental results.

Principal strain in the direction of principal stress σ_1, $e_1 = \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_3)\right]$

Principal strain in the direction of principal stress σ_3, $e_3 = \frac{1}{E}\left[\sigma_3 - \mu(\sigma_1 + \sigma_2)\right]$

According to maximum principal strain theory, the conditions to cause failure are

$$e_1 > \frac{\sigma_{yt}}{E} \; ; \quad \frac{1}{E}\left[\sigma_1 - \mu(\sigma_2 + \sigma_3)\right] > \frac{\sigma_{yt}}{E} \Rightarrow \sigma_1 - \mu(\sigma_2 + \sigma_3) > \sigma_{yt} \qquad \text{or}$$

$$|e_3| > \frac{\sigma_{yc}}{E} \; ; \quad \frac{1}{E}\left[\sigma_3 - \mu(\sigma_1 + \sigma_2)\right] > \frac{\sigma_{yc}}{E} \Rightarrow \sigma_3 - \mu(\sigma_1 + \sigma_2) > \sigma_{yc}$$

To prevent failure

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) < \sigma_{yt}$$

$$\sigma_3 - \mu(\sigma_1 + \sigma_2) < \sigma_{yc}$$

At the point of failure

$$\sigma_1 - \mu(\sigma_2 + \sigma_3) = \sigma_{yt}$$

$$\left|\sigma_3 - \mu(\sigma_1 + \sigma_2)\right| = \sigma_{yc}$$

For the design purposes

$$\sigma_3 - \mu(\sigma_1 + \sigma_2) = \sigma_t$$

$$\sigma_3 - \mu(\sigma_1 + \sigma_2) = \sigma_c$$

Where σ_t and σ_c are the safe stresses.

Maximum shear stress theory

- This theory is also called Coulomb Guest's or Treasca's theory
- It states that the material will fail when the maximum shear stress ($\tau_{\max}$) in the complex system reaches the value of maximum shear stress in simple tension at the elastic limit.

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{yt}}{2}$$

$$\sigma_1 - \sigma_3 = \sigma_{yt}$$

$$\sigma_1 - \sigma_3 = \sigma_t$$

- This theory gives good correlation with the results of experiments on ductile materials.
- It gives satisfactory results for ductile materials particularly in case of shafts.
- The theory does not give accurate results for the state of stress of pure shear.
- The theory is not applicable in the case where the state of stress consists of triaxial tensile stresses of nearly equal magnitude.

4. Maximum strain energy theory

- This theory was proposed by Beltrani-Haigh.
- It states that the failure of a material occurs when the total strain energy in the material reaches the total strain energy of the material at the elastic limit in simple tension.

In a 3D stress system, the strain energy per unit volume is given by

$$U = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right]$$

At the point of failure

$$\frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] = \frac{\sigma_y^2}{2E}$$

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \sigma_y^2$$

In $2D$ stress system ($\sigma_2 = 0$), the above equation reduce to

$$\sigma_1^2 + \sigma_3^2 - 2\mu\sigma_1\sigma_3 = \sigma_y^2$$

For the design

$$\sigma_1^2 + \sigma_3^2 - 2\mu\sigma_1\sigma_3 \le \sigma^2$$

- It is applicable for ductile materials particularly in case of pressure vessel.
- The theory does not applicable to materials for which σ_{yt} is different from σ_{yc}
- The theory does not give results exactly equal to the experimental results even for ductile materials

If a body is subjected to high hydrostatic pressure equal compressive stress on all the three faces, then based on the strain energy theory, the maximum compressive stress will be

$$\sigma = \frac{\sigma_y}{\sqrt{3(1-\mu)}}$$

Yield stress of the material $= \sigma_y$

Strain energy stored, $U = \frac{1}{2}.\sigma_1.e_1 + \frac{1}{2}\sigma_2.e_2 + \frac{1}{2}.\sigma_3.e_3$

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma$$

$$e_1 = e_2 = e_3 = \frac{\sigma}{E} - 2\mu\frac{\sigma}{E} = \frac{\sigma}{E}(1-2\mu)$$

$$U = 3.\sigma.\frac{\sigma}{E}(1-2\mu) = \frac{3\sigma^2}{E}(1-2\mu)$$

Also $U =$ stress $\times$ strain

$$= \sigma_y.\frac{\sigma_y}{E} = \frac{\sigma_y^2}{E}$$

$$\frac{3\sigma^2}{E}(1-2\mu) = \frac{\sigma_y^2}{E} \Rightarrow \quad \sigma = \frac{\sigma_y}{\sqrt{3(1-\mu)}}$$

5. Maximum shear strain energy theory
or
Distortion energy theory

- This theory was proposed by Von Mises-Henky
- It states that the elastic failure occurs when the shear strain energy per unit volume in the stressed material reaches a value equal to the shear strain energy per unit volume at the elastic limit point in tension.
- The theory gives best results for ductile material particularly in case of pure shear or $\sigma_{yc} = \sigma_{yt}$.

Shear strain energy per unit volume due to principal stresses σ_1, σ_2 and σ_3 ,

$$U_S = \frac{1+\mu}{3E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - (\sigma_1\sigma_2 + \sigma_3\sigma_2 + \sigma_1\sigma_3)\right]$$

$$U_S = \frac{1+\mu}{6E}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

$$U_S = \frac{1}{12G}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]$$

For the simple tension at the elastic limit point, $(\sigma_1 = \sigma_{yt}, \sigma_2 = \sigma_3 = 0)$, the shear strain energy per unit volume is given by

$$U'_S = \frac{1}{12G}\left[(\sigma_{yt} - 0)^2 + (0-0)^2 + (0-\sigma_{yt})^2\right] = \frac{1}{12G}.2\sigma_{yt}^2$$

Equating the two strain energies, $(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2.\sigma_t^2$

In $2D$ stress system ($\sigma_2 = 0$), the above equation reduce to

$$\sigma_1^2 + \sigma_3^2 - \sigma_1\sigma_3 = \sigma_t^2$$

Design conditions for various failure theory

Failure theory	Proposed by	Condition for design
Maximum principal stress theory	Rankine, Lame	$\sigma_1 \leq \sigma$
Maximum principal strain theory	Saint Venants	$\frac{1}{E}(\sigma_1 - \mu\sigma_2) \leq \frac{\sigma_y}{E}$
Maximum shear stress theory	Coulomb Guest, Tresca	$\frac{\sigma_1 - \sigma_2}{2} \leq \frac{\sigma_y}{2}$
Maximum strain energy theory	Beltrami-Haigh	$(\sigma_1^2 + \sigma_2^2 - 2\mu\sigma_1\sigma_2) \leq \sigma_y^2$
Distortion energy theory	Huber-Henky-Von Mises	$(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) \leq \sigma_y^2$

- When one of the principal stresses at a point is large in comparison to the other, all the failure theories gives nearly the same result.
- When a member is subjected to uni-axial tension, all the failure theories gives the same result.

GATE
PREVIOUS QUESTIONS
AND
SOLUTIONS

01. A small element at the critical section of a component is in a bi-axial state of stress with the two principle stresses being 360 MPa and 140 MPa. The maximum working stress according to distortion energy theory is: GATE ME 1997

a. 220 MPa b. 110 MPa **c.** 314 MPa d. 330 MPa

01. C.

σ_1 : Major principal stress = 360 MPa

σ_2 : Minor principal stress = 140 MPa

f : working stress in the element according to distortion energy theory

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 \geq f^2$$

$$360^2 + 140^2 - 360 \times 140 \geq f^2$$

$$f = 314.3 \text{ N/mm}^2$$

02. According to Von-Mises distortion energy theory, the distortion energy under three dimensional stress state is represented by GATE ME 2006

a. $\frac{1}{2E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-2\mu(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

b. $\frac{1-2\mu}{6E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2+2(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

c. $\frac{1+\mu}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

d. $\frac{1}{3E}\left[\sigma_1^2+\sigma_2^2+\sigma_3^2-\mu(\sigma_1\sigma_2+\sigma_3\sigma_2+\sigma_1\sigma_3)\right]$

04. The homogenous state of stress for a metal part undergoing plastic deformation is

$$T = \begin{pmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & -10 \end{pmatrix}$$

Where the stress component values are in MPa. Using Von-Mises yield criterion, the value of estimated shear yield stress, in MPa is GATE ME 2012

a. 9.50 b. 16.07 c. 28.52 d. 49.41

04. B

04. B

$$\text{State of Stress, } T = \begin{bmatrix} 10 & 5 & 0 \\ 5 & 20 & 0 \\ 0 & 0 & -10 \end{bmatrix}$$

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

$\sigma_x = 10$ MPa, $\sigma_y = 20$ MPa, $\sigma_z = -10$MPa

$\tau_{xy} = 5$MPa, $\tau_{yz} = 0$, $\tau_{zx} = 0$

$$\text{Principal stresses, } \sigma_{1,3} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4.\tau_{xy}^2}$$

$$\sigma_{1,3} = \frac{10+20}{2} \pm \frac{1}{2}\sqrt{(10-20)^2 + 4\times 5^2} = 15 \pm \frac{1}{2}\sqrt{100+100} = 15 \pm 7.07$$

$\sigma_1 = 15 + 7.07 = 22.07$ MPa

$\sigma_2 = 15 - 7.07 = 7.93$ MPa

σ_{yt} : Yield stress in tension

σ_{ys} : Yield stress in shear

According to Von-Mises yield criterion,

$$(\sigma_{yt})^2 \geq \frac{1}{2}\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]$$

$$(\sigma_{yt})^2 \geq \frac{1}{2}\left[(22.07-7.93)^2+(7.93+10)^2+(-10-22.07)^2\right]$$

$$(\sigma_{yt})^2 \geq 774.9 \Rightarrow \sigma_{yt} \geq 27.84\,\text{MPa}$$

$$\sigma_{ys}=\frac{\sigma_{yt}}{\sqrt{3}}=\frac{27.84}{\sqrt{3}}=16.07\,\text{MPa}$$

05. A machine element is subjected to the following bi-axial state of stress: $\sigma_x = 80$; $\sigma_y = 20$; $\tau_{xy} = 40$. If the shear strength of the material is 100 MPa, the factor of safety as per Tresca's maximum shear stress theory is GATE ME 2015

a. 1.0 **b**. 2.0 c. 2.5 d. 3.3

05. B

Biaxial state of stress for an element:

$\sigma_x = 80\,\text{MPa}$, $\sigma_y = 20\,\text{MPa}$ and $\tau_{xy} = 40\,\text{MPa}$

Shear strength of the material, $\tau = 100\,\text{MPa}$

Maximum shear stress induced in an element, $\tau_{\max} = \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$$= \frac{1}{2}\sqrt{(80-20)^2 + 4(40)^2} = 50\text{ MPa}$$

Factor of safety, $F = \dfrac{\text{shear strength}}{\text{shear stress induced}} = \dfrac{100}{50} = 2$

06. The uniaxial yield stress of a materials is 300 MPa. According to von Mises criterion, the shear yield stress (in MPa) of the material is GATE ME 2015

Ans: 171 to 175

06. 173.2

Uniaxial yield stress of material, $\sigma_{yt} = 300\,\text{MPa}$

Shear yield stress, $\sigma_{ys} = \dfrac{\sigma_{yt}}{\sqrt{3}} = \dfrac{300}{\sqrt{3}} = 173.2$ MPa

07. The principal stresses t a point in a critical section of a machine component are $\sigma_1 = 60$ MPa, $\sigma_2 = 5$ MPa and $\sigma_3 = -40$ MPa. For the material of the component, the tensile yield strength is $\sigma_y = 200\,\text{MPa}$. According to the maximum shear stress theory, the factor of safety is GATE ME 2017

a.1.67 **b.** 2.00 b. 3.60 d. 4.00

07. B

Principal stresses at a point in machine component

$\sigma_1 = 60\,\text{MPa}$, $\sigma_2 = 5\,\text{MPa}$ and $\sigma_3 = -40\,\text{MPa}$

Tensile yield strength, $\sigma_{yt} = 200\,\text{MPa}$

Permissible shear stress, $\tau = \dfrac{\sigma_{yt}}{2} = \dfrac{200}{2} = 100$ MPa

Maximum shear stress, $\tau_{\max} = \dfrac{\sigma_1 - \sigma_3}{2}$

$$= \frac{60-(-40)}{2} = 50 \text{ MPa}$$

Factor of safety, $F = \dfrac{\text{Permissible shear stress}}{\text{Maximum shear stress induced}} = \dfrac{100}{5} = 2$

42. At a critical point in a component, the state of stress is given as $\sigma_{xx} = 100\,MPa$, $\sigma_{xy} = 220\,MPa$, $\sigma_{xy} = \sigma_{yx} = 80\,MPa$ and all other stress components are zero. The yield strength of the material is 468 MPa. The factor of safety on the basis of maximum shear stress theory is…. (round off to one decimal place).

GATE ME 2019

42. 1.8

State of stress at a critical point:

σ_x =100 MPa, σ_y =220MPa and τ =80 MPa

Yield strength of the material, σ_y =468 MPa

Factor of safety, $F = ?$

Principal stresses are, $\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau^2}$

$$\sigma_{1,2} = \frac{100+220}{2} \pm \frac{1}{2}\sqrt{(100-220)^2 + 4 \times 80^2}$$

$$= 160 \pm 100$$

$$\sigma_1 = 160+100 = 260\text{MPa}$$

$$\sigma_2 = 160-100 = 60\text{MPa}$$

Maximum shear stress, $\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{260-60}{2} = 100\text{MPa}$

According to maximum shear stress theory,

$$\tau_{\max} = \frac{\sigma_y}{2F} = \max\left\{\frac{\sigma_1 - \sigma_2}{2}, \frac{\sigma_1}{2} \text{ and } \frac{\sigma_2}{2}\right\}$$

$$\frac{468}{2 \times F} = \frac{260}{2} \Rightarrow F = 1.8$$