

## UNIT - IV

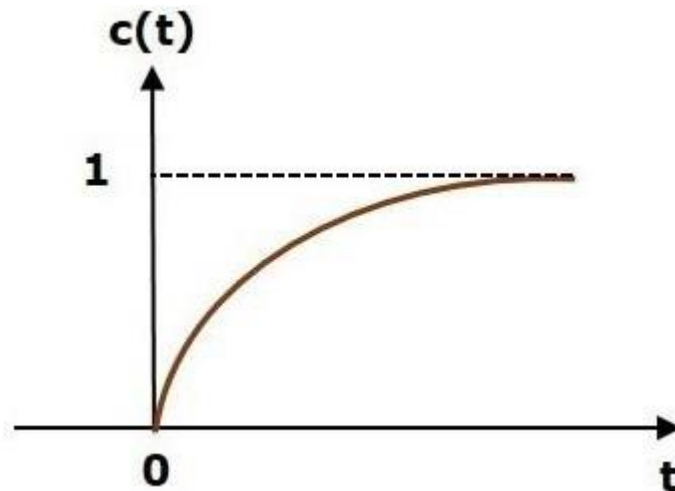
### STABILITY ANALYSIS IN S-DOMAIN

Stability is an important concept. In this chapter, let us discuss the stability of system and types of systems based on stability.

What is Stability?

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A **stable system** produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of  $t$  including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systems based on Stability

We can classify the systems based on stability as follows.

- ☐ Absolutely stable system
- ☐ Conditionally stable system
- ☐ Marginally stable system

### Absolutely Stable System

If the system is stable for all the range of system component values, then it is known as the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of '**s**' plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the 's' plane.

### Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as **conditionally stable system**.

### Marginally Stable System

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

In this chapter, let us discuss the stability analysis in the '**s**' domain using the Routh-Hurwitz stability criterion. In this criterion, we require the characteristic equation to find the stability of the closed loop control systems.

### Routh-Hurwitz Stability Criterion

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

### Necessary Condition for Routh-Hurwitz Stability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Consider the characteristic equation of the order 'n' is -

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s^1 + a_ns^0 = 0$$

Note that, there should not be any term missing in the  $n^{\text{th}}$  order characteristic equation. This means that the  $n^{\text{th}}$  order characteristic equation should not have any coefficient that is of zero value.

### Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

### Routh Array Method

If all the roots of the characteristic equation exist to the left half of the 's' plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the 's' plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.

So, to overcome this problem there we have the **Routh array method**. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the 's' plane and the control system is unstable.

Follow this procedure for forming the Routh table.

- ② Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of  $s^n$  and continue up to the coefficient of  $s^0$ .
- ② Fill the remaining rows of the Routh array with the elements as mentioned in the table below. Continue this process till you get the first column element of **row**  $s_0s_0$  is an. Here, an is the coefficient of  $s^0$  in the characteristic polynomial.

**Note** – If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

The following table shows the Routh array of the  $n^{\text{th}}$  order characteristic polynomial.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s^1 + a_n s^0$$

$s^n$	$a_0$	$a_2$	$a_4$	$a_6$	...	...
$s^{n-1}$	$a_1$	$a_3$	$a_5$	$a_7$	...	...
$s^{n-2}$	$b_1$ $= \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2$ $= \frac{a_1 a_4 - a_5 a_0}{a_1}$	$b_3$ $= \frac{a_1 a_6 - a_7 a_0}{a_1}$	...	...	...
$s^{n-3}$	$c_1$ $= \frac{b_1 a_3 - b_2 a_1}{b_1}$	$c_2$ $= \frac{b_1 a_5 - b_2 a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
$s^1$	$\vdots$	$\vdots$				
$s^0$	$a_n$					

### Example

Let us find the stability of the control system having characteristic equation,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

**Step 1** – Verify the necessary condition for the Routh-Hurwitz

stability. All the coefficients of the characteristic polynomial,

$s^4 + 3s^3 + 3s^2 + 2s + 1$  are positive. So, the control system satisfies the necessary condition.

**Step 2** – Form the Routh array for the given characteristic polynomial.

$s^4$	1	3	1
$s^3$	3	2	
$s^2$	$\frac{(3 \times 3) - (2 \times 1)}{3} = \frac{7}{3}$	$\frac{(3 \times 1) - (0 \times 1)}{3} = \frac{3}{3}$ $= 1$	
$s^1$	$\frac{\left(\frac{7}{3} \times 2\right) - (1 \times 3)}{\frac{7}{3}}$ $= \frac{5}{7}$		
$s^0$	1		

**Step 3** – Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive. There is no sign change in the first column of the Routh array. So, the control system is stable.

### Special Cases of Routh Array

We may come across two types of situations, while forming the Routh table. It is difficult to complete the Routh table from these two situations.

The two special cases are –

- ☐ The first element of any row of the Routh's array is zero.
- ☐ All the elements of any row of the Routh's array are zero.

Let us now discuss how to overcome the difficulty in these two cases, one by one.

### First Element of any row of the Routh's array is zero

If any row of the Routh's array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer,  $\epsilon$ . And then continue the process of completing the Routh's table. Now, find the number of sign changes in the first column of the Routh's table by substituting  $\epsilon \rightarrow 0$ .

### Example

Let us find the stability of the control system having characteristic equation,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

**Step 1** – Verify the necessary condition for the Routh-Hurwitz

stability. All the coefficients of the characteristic polynomial,

$$s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

are positive. So, the control system satisfied the necessary condition.

**Step 2** – Form the Routh array for the given characteristic polynomial.

$s^4$	1	1	1
$s^3$	$\geq 1$	$\geq 1$	
$s^2$	$\frac{(1 \times 1) - (1 \times 1)}{1} = 0$	$\frac{(1 \times 1) - (0 \times 1)}{1} = 1$	
$s^1$			
$s^0$			

The row  $s^3$  elements have 2 as the common factor. So, all these elements are divided by 2.

**Special case (i)** – Only the first element of row  $s^2$  is zero. So, replace it by  $\epsilon$  and continue the process of completing the Routh table.

$s^4$	1	1	1
$s^3$	1	1	
$s^2$	$\epsilon$	1	
$s^1$	$\frac{(\epsilon \times 1) - (1 \times 1)}{\epsilon} = \frac{\epsilon - 1}{\epsilon}$		
$s^0$	1		

**Step 3** – Verify the sufficient condition for the Routh-Hurwitz

stability. As  $\epsilon$  tends to zero, the Routh table becomes like this.

$s^4$	1	1	1
$s^3$	1	1	
$s^2$	0	1	
$s^1$	$-\infty$		
$s^0$	1		

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

**All the Elements of any row of the Routh's array are zero**

In this case, follow these two steps –

- ☐ Write the auxiliary equation,  $A(s)$  of the row, which is just above the row of zeros.
- ☐ Differentiate the auxiliary equation,  $A(s)$  with respect to  $s$ . Fill the row of zeros with these coefficients.

### Example

Let us find the stability of the control system having characteristic equation,

$$s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

**Step 1** – Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the given characteristic polynomial are positive. So, the control system satisfied the necessary condition.

**Step 2** – Form the Routh array for the given characteristic polynomial.

$s^5$	1	1	1
$s^4$	$\exists 1$	$\exists 1$	$\exists 1$
$s^3$	$\frac{(1 \times 1) - (1 \times 1)}{1} = 0$	$\frac{(1 \times 1) - (1 \times 1)}{1} = 0$	
$s^2$			
$s^1$			
$s^0$			

The row  $s^4$  elements have the common factor of 3. So, all these elements are divided by 3.

**Special case (ii)** – All the elements of row  $s^3$  are zero. So, write the auxiliary equation,  $A(s)$  of the row  $s^4$ .

$$A(s) = s^4 + s^2 + 1$$

$$\frac{dA(s)}{ds} = 4s^3 + 2s$$

Place these coefficients in row  $s^3$ .

$s^5$	1	1	1
$s^4$	1	1	1
$s^3$	4 2	$\cong 1$	
$s^2$	$\frac{(2 \times 1) - (1 \times 1)}{2} = 0.5$	$\frac{(2 \times 1) - (0 \times 1)}{2} = 1$	
$s^1$	$\frac{(0.5 \times 1) - (1 \times 2)}{0.5} = \frac{-1.5}{0.5}$ $= -3$		
$s^0$	1		

**Step 3** – Verify the sufficient condition for the Routh-Hurwitz stability.

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

In the Routh-Hurwitz stability criterion, we can know whether the closed loop poles are in on left half of the 's' plane or on the right half of the 's' plane or on an imaginary axis. So, we can't find the nature of the control system. To overcome this limitation, there is a technique known as the root locus.

### Root locus Technique

In the root locus diagram, we can observe the path of the closed loop poles. Hence, we can identify the nature of the control system. In this technique, we will use an open loop transfer function to know the stability of the closed loop control system.

## Basics of Root Locus

The Root locus is the locus of the roots of the characteristic equation by varying system gain  $K$  from zero to infinity.

We know that, the characteristic equation of the closed loop control system is

$$1 + G(s)H(s) = 0$$

We can represent  $G(s)H(s)$  as

$$G(s)H(s) = K \frac{N(s)}{D(s)}$$

Where,

- $K$  represents the multiplying factor
- $N(s)$  represents the numerator term having (factored)  $n^{\text{th}}$  order polynomial of 's'.
- $D(s)$  represents the denominator term having (factored)  $m^{\text{th}}$  order polynomial of 's'.

Substitute,  $G(s)H(s)$  value in the characteristic equation.

$$1 + k \frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

### Case 1 – $K = 0$

If  $K = 0$ , then  $D(s) = 0$ .

That means, the closed loop poles are equal to open loop poles when  $K$  is zero.

### Case 2 – $K = \infty$

Re-write the above characteristic equation as

$$K \left( \frac{1}{K} + \frac{N(s)}{D(s)} \right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Substitute,  $K = \infty$  in the above equation.

$$\frac{1}{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$$

If  $K = \infty$ , then  $N(s) = 0$ . It means the closed loop poles are equal to the open loop zeros when  $K$  is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

### Angle Condition and Magnitude Condition

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of  $K$  for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$$\begin{aligned} 1 + G(s)H(s) &= 0 \\ \Rightarrow G(s)H(s) &= -1 + j0 \end{aligned}$$

The **phase angle** of  $G(s)H(s)$  is

$$\angle G(s)H(s) = \tan^{-1} \left( \frac{0}{-1} \right) = (2n + 1)\pi$$

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of  $180^\circ$ .

Magnitude of  $G(s)H(s)G(s)H(s)$  is -

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

The **root locus** is a graphical representation in s-domain and it is symmetrical about the real axis. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs. In this chapter, let us discuss how to construct (draw) the root locus.

Rules for Construction of Root Locus

Follow these rules for constructing a root locus.

**Rule 1** – Locate the open loop poles and zeros in the 's' plane.

**Rule 2** – Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$$N=P \text{ if } P \geq Z$$

$$N=Z \text{ if } P < Z$$

**Rule 3** – Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of  $180^\circ$ , then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

**Rule 4** – Find the centroid and the angle of asymptotes.

- ☐ If  $P=Z$ , then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- ☐ If  $P>Z$ , then  $Z$  number of root locus branches start at finite open loop poles and end at finite open loop zeros and  $P-Z$  number of root locus branches start at finite open loop poles and end at infinite open loop zeros.

- ☐ If  $P < Z$ , then  $P$  number of root locus branches start at finite open loop poles and end at finite open loop zeros and  $Z - P$  number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when  $P \neq Z$ . Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid  $\alpha$**  by using this formula,

$$\alpha = \frac{\sum \text{Real part of finite open loop poles} - \sum \text{Real part of finite open loop zeros}}{P - Z}$$

The formula for the angle of **asymptotes  $\theta$**  is

$$\theta = \frac{(2q + 1)180^\circ}{P - Z}$$

Where,

$$q = 0, 1, 2, \dots, (P - Z) - 1$$

**Rule 5** – Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- ☐ If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- ☐ Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- ☐ Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

**Rule 6** – Find Break-away and Break-in points.

- ☐ If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.

- ☐ If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

**Note** – Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- ☐ Write K in terms of s from the characteristic equation  $1+G(s)H(s)=0$ .
- ☐ Differentiate K with respect to s and make it equal to zero. Substitute these values of s in the above equation.
- ☐ The values of s for which the K value is positive are the **break points**.

**Rule 7** – Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure**  $\phi_d$  is

$$\phi_d = 180^\circ - \phi$$

The formula for the **angle of arrival**  $\phi_a$  is

$$\phi_a = 180^\circ + \phi$$

Where,

$$\phi = \sum \phi_P - \sum \phi_Z$$

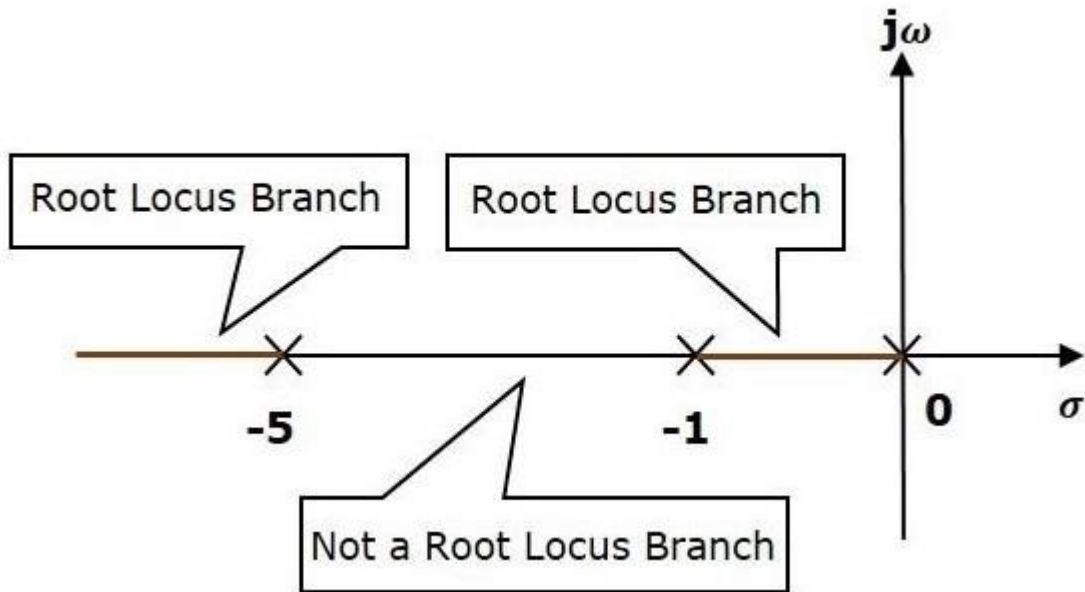
Example

Let us now draw the root locus of the control system having open loop transfer function,

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

**Step 1** – The given open loop transfer function has three poles at  $s = 0$ ,  $s = -1$ ,  $s = -5$ . It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function.

$$N=P=3$$



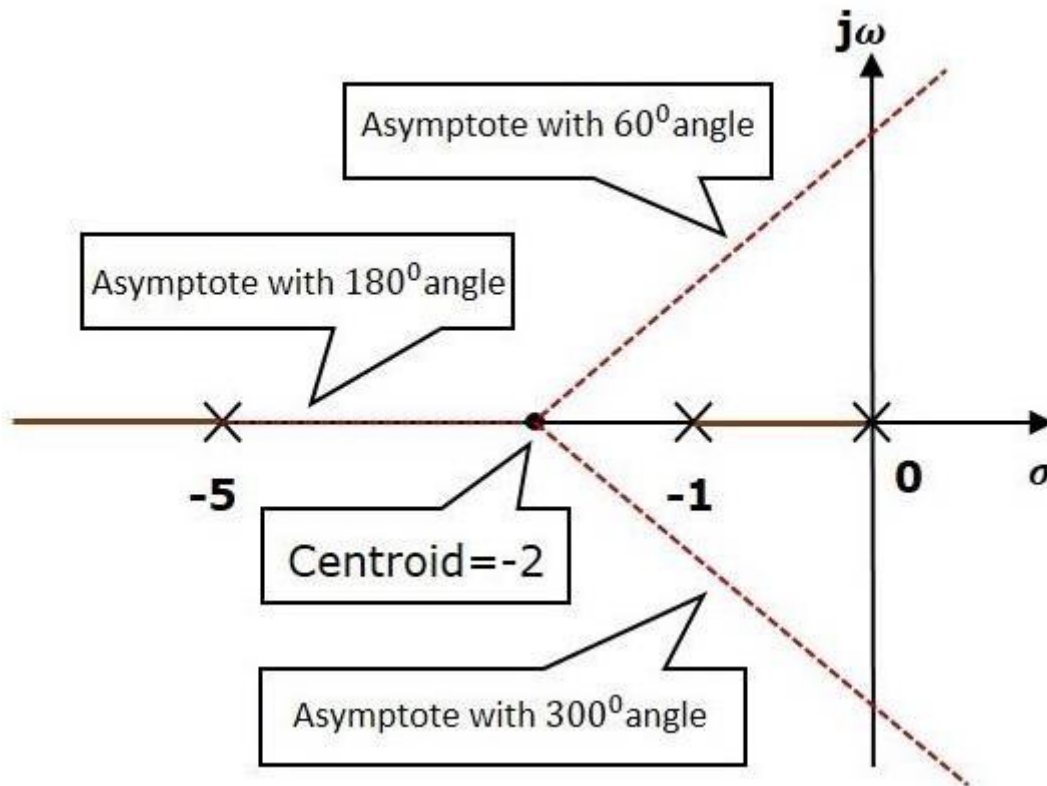
The three poles are located as shown in the above figure. The line segment between  $s = -1$ , and  $s = 0$  is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of  $s = -5$ .

**Step 2** – We will get the values of the centroid and the angle of asymptotes by using the given formulae.

Centroid

The angle of asymptotes are  $\theta = 60^\circ, 180^\circ$  and  $300^\circ$ .

The centroid and three asymptotes are shown in the following figure.



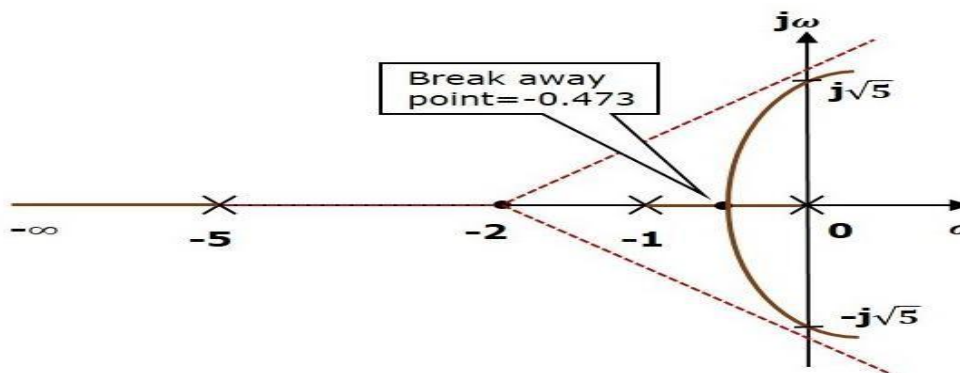
**Step 3** – Since two asymptotes have the angles of  $60^\circ$  and  $300^\circ$ , two root locus branches intersect the imaginary axis. By using the Routh array method and special case(ii),

$$j\sqrt{5} \quad -j\sqrt{5}$$

the root locus branches intersect the imaginary axis at

and  $j\sqrt{5}$  and  $-j\sqrt{5}$ . There will be one break-away point on the real axis root locus branch between the poles  $s = -1$  and  $s = 0$ . By following the procedure given for the calculation of break-away point, we will get it as  $s = -0.473$ .

The root locus diagram for the given control system is shown in the following figure.



In this way, you can draw the root locus diagram of any control system and observe the movement of poles of the closed loop transfer function.

From the root locus diagrams, we can know the range of K values for different types of damping.

#### Effects of Adding Open Loop Poles and Zeros on Root Locus

The root locus can be shifted in '**s**' plane by adding the open loop poles and the open loop zeros.

- ☒ If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of 's' plane. Because of this, the damping ratio  $\delta$  decreases. Which implies, damped frequency  $\omega_d$  increases and the time domain specifications like delay time  $t_d$ , rise time  $t_r$  and peak time  $t_p$  decrease. But, it effects the system stability.
- ☒ If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of 's' plane. So, it will increase the control system stability. In this case, the damping ratio  $\delta$  increases. Which implies, damped frequency  $\omega_d$  decreases and the time domain specifications like delay time  $t_d$ , rise time  $t_r$  and peak time  $t_p$  increase.

So, based on the requirement, we can include (add) the open loop poles or zeros to the transfer function.

Example

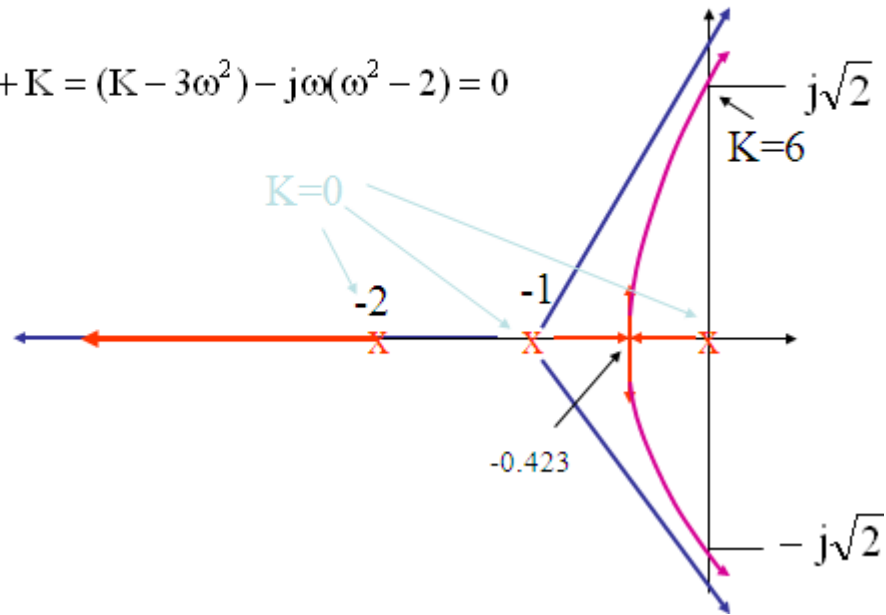
$$s^3 + 3s^2 + 2s + K = 0$$

$$s = j\omega$$

$$-j\omega^3 - 3\omega^2 + 2j\omega + K = (K - 3\omega^2) - j\omega(\omega^2 - 2) = 0$$

$$\omega^2 = 2$$

$$K = 3\omega^2 = 6$$



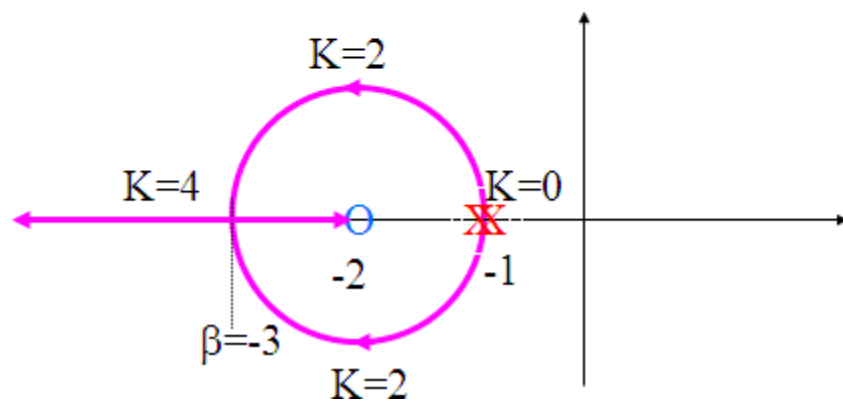
Example

$$GH(s) = \frac{K(s+2)}{(s+1)^2}$$

Two poles at  $-1$   
 One zero at  $-2$   
 One asymptote at  $180^\circ$   
 Break-in point at  $-3$

$$\frac{2}{\beta+1} = \frac{1}{\beta+2}$$

$$2\beta+4 = \beta+1$$



Example Why a circle ?

Characteristic equation  $s^2 + s(2 + K) + 2K + 1 = 0$

For  $K < 4$

$$s_{1,2} = \frac{-(2 + K) \pm j\sqrt{K(4 - K)}}{2}$$

For  $K > 4$

$$s_{1,2} = \frac{-(2 + K) \pm \sqrt{K(K - 4)}}{2}$$

Change of origin  $s_{1,2} + 2 = \frac{-(-2 + K) \pm j\sqrt{K(4 - K)}}{2}$

$$4m = (K - 2)^2 + K(4 - K) = K^2 - 4K + 4 + 4K - K^2$$

$$m = 1$$

