

UNIT-III

TIME RESPONSE ANALYSIS

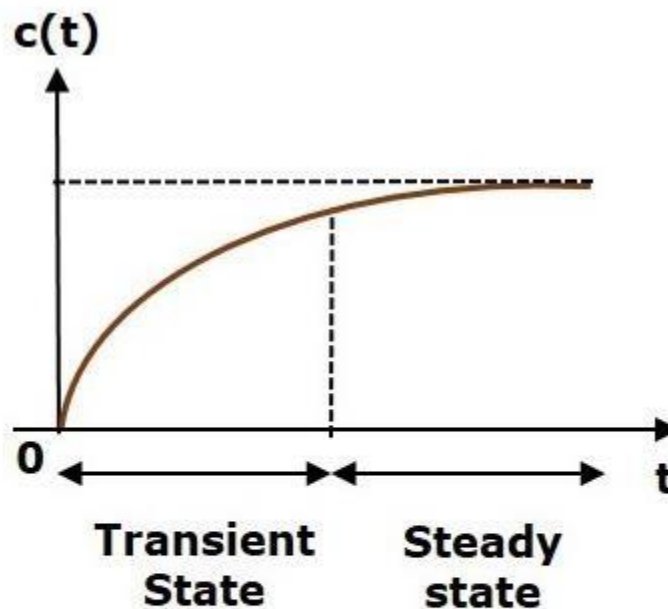
We can analyze the response of the control systems in both the time domain and the frequency domain. We will discuss frequency response analysis of control systems in later chapters. Let us now discuss about the time response analysis of control systems.

What is Time Response?

If the output of control system for an input varies with respect to time, then it is called the **time response** of the control system. The time response consists of two parts.

- ☐ Transient response
- ☐ Steady state response

The response of control system in time domain is shown in the following figure.



Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response $c(t)$ as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

- ☐ $c_{tr}(t)$ is the transient response
- ☐ $c_{ss}(t)$ is the steady state response

Transient Response

After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as **transient response**.

The transient response will be zero for large values of 't'. Ideally, this value of 't' is infinity and practically, it is five times constant.

Mathematically, we can write it as

$$\lim_{t \rightarrow \infty} c_{tr}(t) = 0$$

Steady state Response

The part of the time response that remains even after the transient response has zero value for large values of 't' is known as **steady state response**. This means, the transient response will be zero even during the steady state.

Example

Let us find the transient and steady state terms of the time response of the control system

$$c(t) = 10 + 5e^{-t}$$

Here, the second term $5e^{-t}$ will be zero as **t** denotes infinity. So, this is the **transient term**. And the first term 10 remains even as **t** approaches infinity. So, this is the **steady state term**.

Standard Test Signals

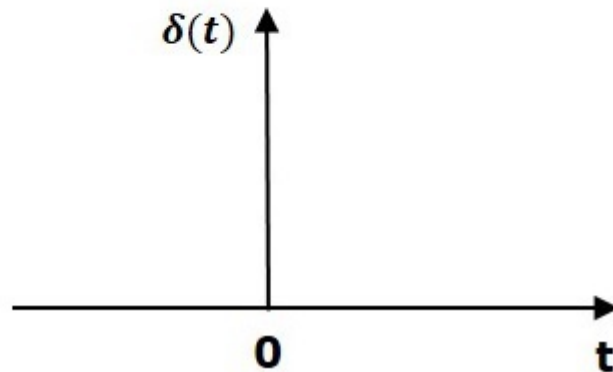
The standard test signals are impulse, step, ramp and parabolic. These signals are used to know the performance of the control systems using time response of the output.

Unit Impulse Signal

A unit impulse signal, $\delta(t)$ is defined as

$$\delta(t) = 0 \text{ for } t \neq 0$$
$$\text{and } \int_{0^-}^{0^+} \delta(t) dt = 1$$

The following figure shows unit impulse signal.



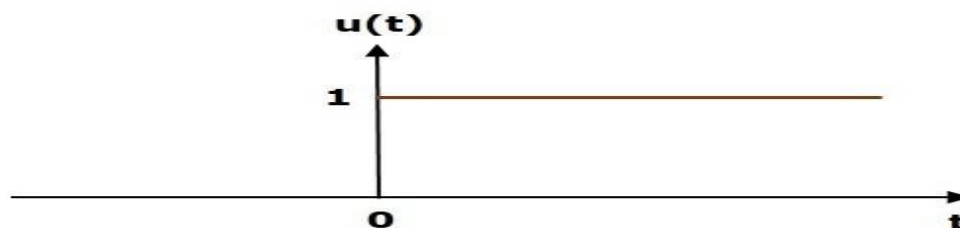
So, the unit impulse signal exists only at t is equal to zero. The area of this signal under small interval of time around t is equal to zero is one. The value of unit impulse signal is zero for all other values of t .

Unit Step Signal

A unit step signal, $u(t)$ is defined as

$$u(t) = 1; t \geq 0$$
$$= 0; t < 0$$

Following figure shows unit step signal.



So, the unit step signal exists for all positive values of 't' including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of 't'.

Unit Ramp Signal

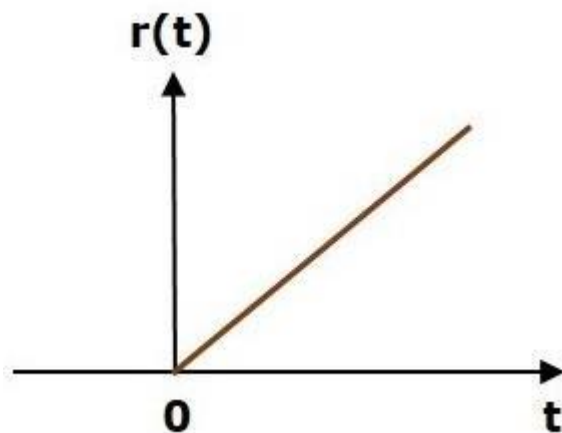
A unit ramp signal, $r(t)$ is defined as

$$r(t) = t; t \geq 0$$
$$= 0; t < 0$$

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

$$r(t) = tu(t)$$

Following figure shows unit ramp signal.



So, the unit ramp signal exists for all positive values of 't' including zero. And its value increases linearly with respect to 't' during this interval. The value of unit ramp signal is zero for all negative values of 't'.

Unit Parabolic Signal

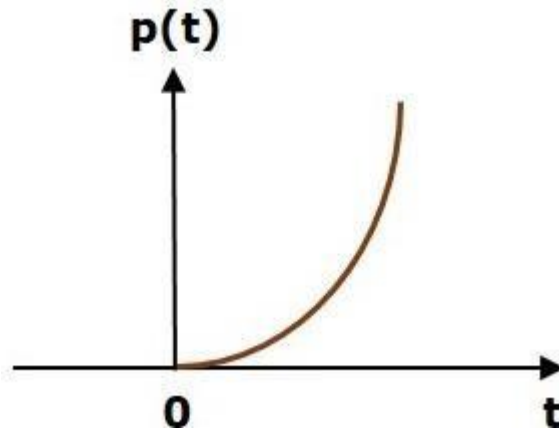
A unit parabolic signal, $p(t)$ is defined as,

$$p(t) = \frac{t^2}{2}; t \geq 0$$
$$= 0; t < 0$$

We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

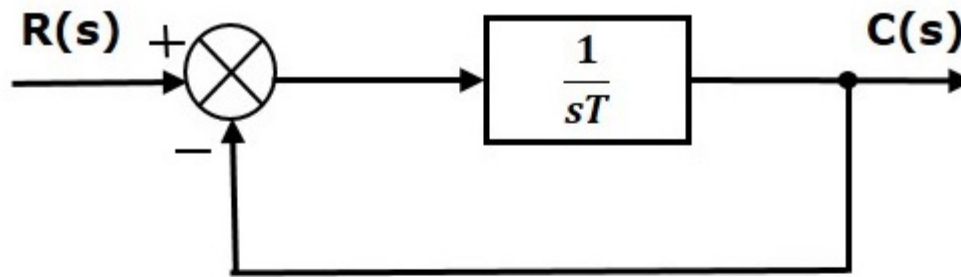
$$p(t) = \frac{t^2}{2}u(t)$$

The following figure shows the unit parabolic signal.



So, the unit parabolic signal exists for all the positive values of t including zero. And its value increases non-linearly with respect to t during this interval. The value of the unit parabolic signal is zero for all the negative values of t .

In this chapter, let us discuss the time response of the first order system. Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, $1/sT$ is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT + 1} \right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal $c(t)$,
- **R(s)** is the Laplace transform of the input signal $r(t)$, and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1} \right) R(s)$
- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to the first order system.

So, $r(t) = \delta(t)$

Apply Laplace transform on both the

sides. $R(s) = 1$

Consider the equation, $C(s) = \left(\frac{1}{sT+1} \right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT + 1} \right) (1) = \frac{1}{sT + 1}$$

Rearrange the above equation in one of the standard forms of Laplace transforms.

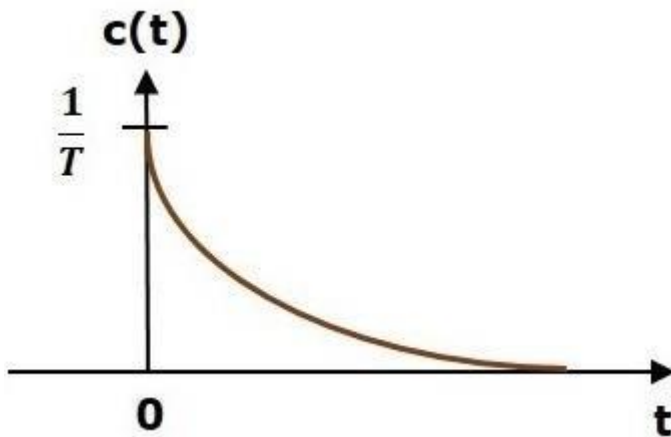
$$C(s) = \frac{1}{T \left(s + \frac{1}{T} \right)} \Rightarrow C(s) = \frac{1}{T} \left(\frac{1}{s + \frac{1}{T}} \right)$$

Applying Inverse Laplace Transform on both the sides,

$$c(t) = \frac{1}{T} e^{\left(-\frac{t}{T}\right)} u(t)$$

$$c(t) = \frac{1}{T} e^{\left(-\frac{t}{T}\right)} u(t)$$

The unit impulse response is shown in the following figure.



The **unit impulse response**, $c(t)$ is an exponential decaying signal for positive values of ' t ' and it is zero for negative values of ' t '.

Step Response of First Order System

Consider the **unit step signal** as an input to first order system. So, $r(t)=u(t)$

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right) \left(\frac{1}{s}\right) = \frac{1}{s(sT+1)}$$

Do partial fractions of C(s).

$$C(s) = \frac{1}{s(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s(sT+1)} = \frac{A(sT+1) + Bs}{s(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs$$

By equating the constant terms on both the sides, you will get $A = 1$.

Substitute, $A = 1$ and equate the coefficient of the s terms on both the sides.

$$0 = T + B$$

$$\Rightarrow B = -T$$

Substitute, $A = 1$ and $B = -T$ in partial fraction expansion of C(s)

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s + \frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit step response**, $c(t)$ has both the transient and the steady state terms.

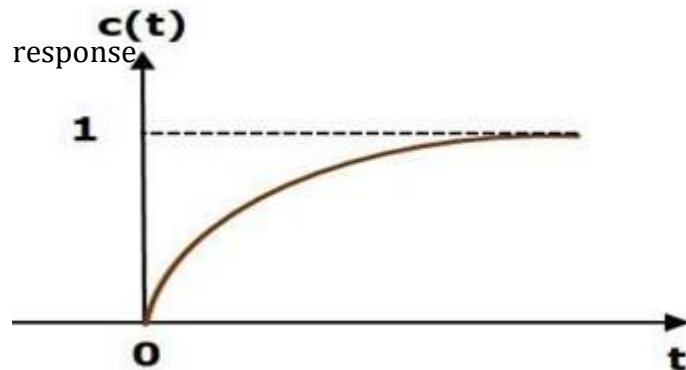
The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit step response

$$c_{ss}(t) = u(t)$$

is - The following figure shows the unit step



The value of the **unit step response**, $c(t)$ is zero at $t = 0$ and for all negative values of t . It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

Ramp Response of First Order System

Consider the **unit ramp signal** as an input to the first order system.

So, $r(t) = t u(t)$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^2}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s^2}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right) \left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT + 1) + Bs(sT + 1) + Cs^2$$

By equating the constant terms on both the sides, you will get $A = 1$.

Substitute, $A = 1$ and equate the coefficient of the s terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Similarly, substitute $B = -T$ and equate the coefficient of s^2 terms on both the sides. You will get $C = T^2$

Substitute $A = 1$, $B = -T$ and $C = T^2$ in the partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{sT + 1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T\left(s + \frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s + \frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(t - T + Te^{-\left(\frac{t}{T}\right)} \right) u(t)$$

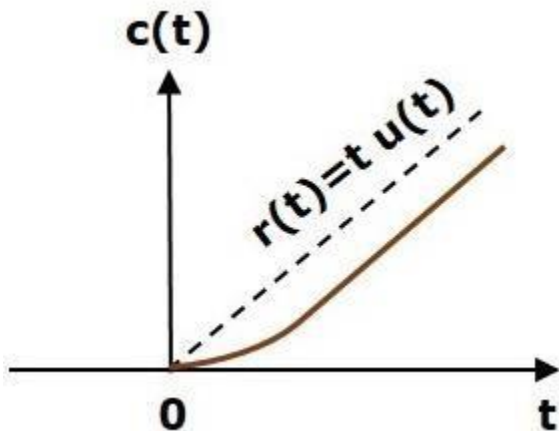
The **unit ramp response**, $c(t)$ has both the transient and the steady state terms. The transient term in the unit ramp response is

$$c_{tr}(t) = Te^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit ramp response is –

$$c_{ss}(t) = (t - T)u(t)$$

The figure below is the unit ramp response:



The **unit ramp response**, $c(t)$ follows the unit ramp input signal for all positive values of t . But, there is a deviation of T units from the input signal.

Parabolic Response of First Order System

Consider the **unit parabolic signal** as an input to the first order system.

$$\text{So, } r(t) = \frac{t^2}{2}u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s^3}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute $R(s) = \frac{1}{s^3}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right) \left(\frac{1}{s^3}\right) = \frac{1}{s^3(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s^3(sT + 1)} = \frac{A}{s^3} + \frac{B}{s^2} + \frac{C}{s} + \frac{D}{sT + 1}$$

After simplifying, you will get the values of A, B, C and D as 1, $-T$, T^2 and $-T^3$ respectively. Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^3}{sT+1} \Rightarrow C(s) = \frac{1}{s^3} - \frac{T}{s^2} + \frac{T^2}{s} - \frac{T^2}{s+\frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(\frac{t^2}{2} - Tt + T^2 - T^2 e^{-\left(\frac{t}{T}\right)} \right) u(t)$$

The **unit parabolic response**, $c(t)$ has both the transient and the steady state terms.

The transient term in the unit parabolic response is

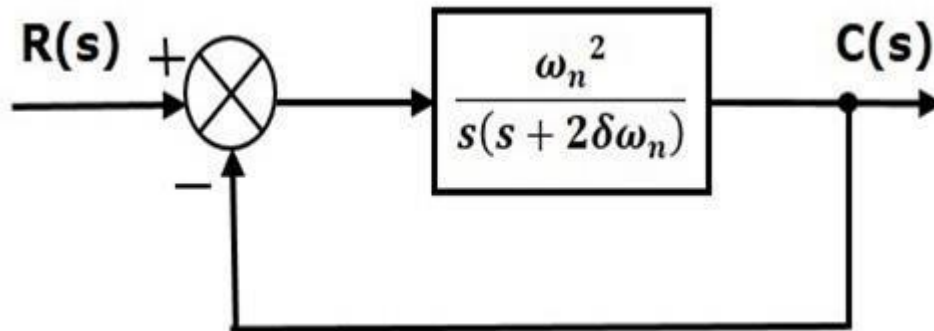
$$C_{tr}(t) = -T^2 e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit parabolic response is

$$C_{ss}(t) = \left(\frac{t^2}{2} - Tt + T^2 \right) u(t)$$

From these responses, we can conclude that the first order control systems are not stable with the ramp and parabolic inputs because these responses go on increasing even at infinite amount of time. The first order control systems are stable with impulse and step inputs because these responses have bounded output. But, the impulse response doesn't have steady state term. So, the step signal is widely used in the time domain for analyzing the control systems from their responses.

In this chapter, let us discuss the time response of second order system. Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\omega_n^2 / s(s+2\delta\omega_n)$ is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1 + \left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\delta\omega_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$
$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1} \quad \text{b}$$

- ☒ The two roots are imaginary when $\delta = 0$.
- ☒ The two roots are real and equal when $\delta = 1$.
- ☒ The two roots are real but not equal when $\delta > 1$.
- ☒ The two roots are complex conjugate when $0 < \delta < 1$.

1. We can write $C(s)$ equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$$

Where,

- ☒ $C(s)$ is the Laplace transform of the output signal, $c(t)$
- ☒ $R(s)$ is the Laplace transform of the input signal, $r(t)$
- ☒ ω_n is the natural frequency
- ☒ δ is the damping ratio.

Follow these steps to get the response (output) of the second order system in the time domain.

Take Laplace transform of the input signal, $r(t)$.

Consider the equation, $C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$

Substitute $R(s)$ value in the above equation.

Do partial fractions of $C(s)$ if required.

Apply inverse Laplace transform to $C(s)$.

Step Response of Second Order System

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Case 1: $\delta = 0$

Substitute, $\delta = 0$ in the transfer function.

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{\omega_n^2}{s^2 + \omega_n^2} \\ \Rightarrow C(s) &= \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \right) R(s)\end{aligned}$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t)) u(t)$$

So, the unit step response of the second order system when $\delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Case 2: $\delta = 1$

Substitute, $\delta = 1$ in the transfer function.

$$\begin{aligned}\frac{C(s)}{R(s)} &= \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2} \\ \Rightarrow C(s) &= \left(\frac{\omega_n^2}{(s + \omega_n)^2} \right) R(s)\end{aligned}$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \omega_n)^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s + \omega_n)^2}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s + \omega_n)^2} = \frac{A}{s} + \frac{B}{s + \omega_n} + \frac{C}{(s + \omega_n)^2}$$

After simplifying, you will get the values of A, B and C as 1, -1 and $-\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

So, the unit step response of the second order system will try to reach the step input in steady state.

Case 3: $0 < \delta < 1$

We can modify the denominator term of the transfer function as follows –

$$\begin{aligned} s^2 + 2\delta\omega_n s + \omega_n^2 &= \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2 \\ &= (s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2) \end{aligned}$$

The transfer function becomes,

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} \\ \Rightarrow C(s) &= \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} \right) R(s) \end{aligned}$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2))}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s((s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2))} = \frac{A}{s} + \frac{Bs + C}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

After simplifying, you will get the values of A, B and C as 1, -1 and $-2\delta\omega_n$ respectively. Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} - \frac{\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + (\omega_n\sqrt{1 - \delta^2})^2} - \frac{\delta}{\sqrt{1 - \delta^2}} \left(\frac{\omega_n\sqrt{1 - \delta^2}}{(s + \delta\omega_n)^2 + (\omega_n\sqrt{1 - \delta^2})^2} \right)$$

Substitute, $\omega_n\sqrt{1 - \delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1 - \delta^2}} \left(\frac{\omega_d}{(s + \delta\omega_n)^2 + \omega_d^2} \right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t} \cos(\omega_d t) - \frac{\delta}{\sqrt{1 - \delta^2}} e^{-\delta\omega_n t} \sin(\omega_d t) \right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} \left((\sqrt{1 - \delta^2}) \cos(\omega_d t) + \delta \sin(\omega_d t) \right) \right) u(t)$$

If $\sqrt{1 - \delta^2} = \sin(\theta)$, then ' δ ' will be $\cos(\theta)$. Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} (\sin(\theta) \cos(\omega_d t) + \cos(\theta) \sin(\omega_d t)) \right) u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} \right) \sin(\omega_d t + \theta) \right) u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Case 4: $\delta > 1$

We can modify the denominator term of the transfer function as follows -

$$\begin{aligned} s^2 + 2\delta\omega_n s + \omega_n^2 &= \{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\} + \omega_n^2 - (\delta\omega_n)^2 \\ &= (s + \delta\omega_n)^2 - \omega_n^2 (\delta^2 - 1) \end{aligned}$$

The transfer function becomes,

$$\begin{aligned} \frac{C(s)}{R(s)} &= \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2 (\delta^2 - 1)} \\ \Rightarrow C(s) &= \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2 (\delta^2 - 1)} \right) R(s) \end{aligned}$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - (\omega_n \sqrt{\delta^2 - 1})^2} \right) \left(\frac{1}{s} \right) = \frac{\omega_n^2}{s(s + \delta\omega_n + \omega_n \sqrt{\delta^2 - 1})(s + \delta\omega_n - \omega_n \sqrt{\delta^2 - 1})}$$

Do partial fractions of $C(s)$.

$$\begin{aligned} C(s) &= \frac{\omega_n^2}{s(s + \delta\omega_n + \omega_n \sqrt{\delta^2 - 1})(s + \delta\omega_n - \omega_n \sqrt{\delta^2 - 1})} \\ &= \frac{A}{s} + \frac{B}{s + \delta\omega_n + \omega_n \sqrt{\delta^2 - 1}} + \frac{C}{s + \delta\omega_n - \omega_n \sqrt{\delta^2 - 1}} \end{aligned}$$

After simplifying, you will get the values of A, B and C as $1, \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}$

and $\frac{-1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}$ respectively. Substitute these values in above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \left(\frac{1}{s + \delta\omega_n + \omega_n \sqrt{\delta^2 - 1}} \right) - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) \left(\frac{1}{s + \delta\omega_n - \omega_n \sqrt{\delta^2 - 1}} \right)$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 + \left(\frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) e^{-(\delta\omega_n + \omega_n \sqrt{\delta^2 - 1})t} - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})} \right) e^{-(\delta\omega_n - \omega_n \sqrt{\delta^2 - 1})t} \right) u(t)$$

Since it is over damped, the unit step response of the second order system when $\delta > 1$ will never reach step input in the steady state.

Impulse Response of Second Order System

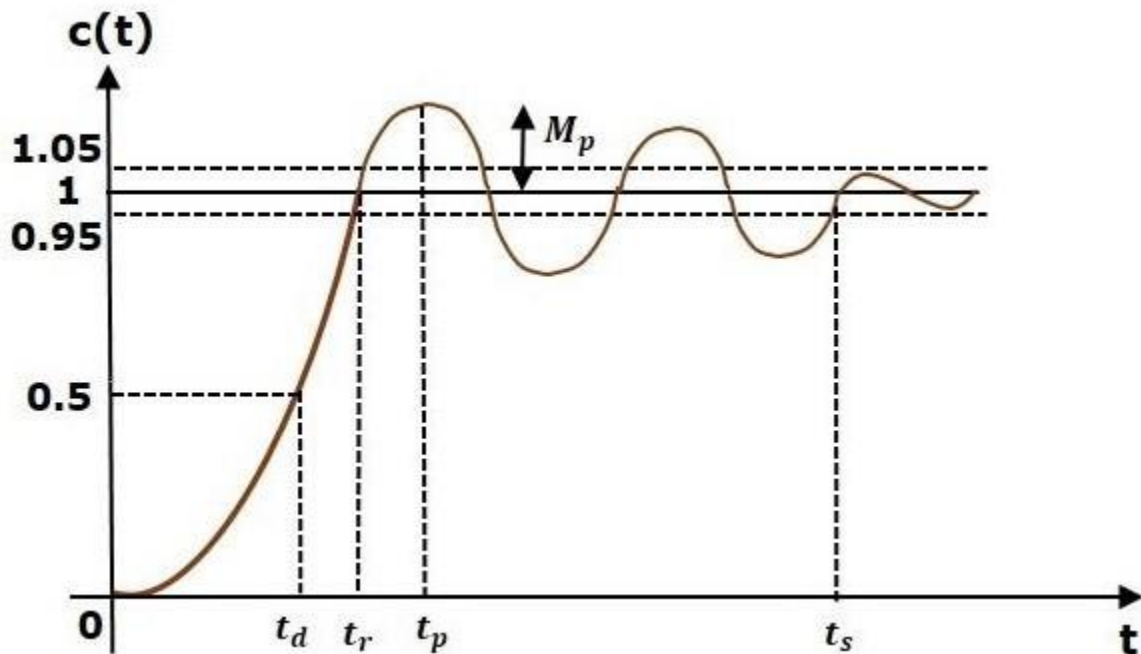
The **impulse response** of the second order system can be obtained by using any one of these two methods.

- ☐ Follow the procedure involved while deriving step response by considering the value of $R(s)$ as 1 instead of $1/s$.
- ☐ Do the differentiation of the step response.

The following table shows the impulse response of the second order system for 4 cases of the damping ratio.

Condition of Damping ratio	Impulse response for $t \geq 0$
$\delta = 0$	$\omega_n \sin(\omega_n t)$
$\delta = 1$	$\omega_n^2 t e^{-\omega_n t}$
$0 < \delta < 1$	$\left(\frac{\omega_n e^{-\delta \omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t)$
$\delta > 1$	$\left(\frac{\omega_n}{2\sqrt{\delta^2-1}} \right) \left(e^{-(\delta\omega_n - \omega_n\sqrt{\delta^2-1})t} - e^{-(\delta\omega_n + \omega_n\sqrt{\delta^2-1})t} \right)$

In this chapter, let us discuss the time domain specifications of the second order system. The step response of the second order system for the underdamped case is shown in the following figure.



All the time domain specifications are represented in this figure. The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by t_d .

Consider the step response of the second order system for $t \geq 0$, when ' δ ' lies between zero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1 - \delta^2}} \right) \sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$\begin{aligned} c(t_d) = 0.5 &= 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1 - \delta^2}} \right) \sin(\omega_d t_d + \theta) \\ \Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1 - \delta^2}} \right) \sin(\omega_d t_d + \theta) &= 0.5 \end{aligned}$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by t_r .

At $t = t_1 = 0$, $c(t) = 0$.

We know that the final value of the step response is one. Therefore, at $t = t_2$, the value of step response is one. Substitute, these values in the following equation.

$$\begin{aligned}
c(t) &= 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta) \\
c(t_2) &= 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_2 + \theta) \\
&\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_2 + \theta) = 0 \\
&\Rightarrow \sin(\omega_d t_2 + \theta) = 0 \\
&\Rightarrow \omega_d t_2 + \theta = \pi \\
&\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}
\end{aligned}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$\begin{aligned}
t_r &= t_2 - t_1 \\
\therefore t_r &= \frac{\pi - \theta}{\omega_d}
\end{aligned}$$

From above equation, we can conclude that the rise time t_r and the damped frequency ω_d are inversely proportional to each other.

Peak Time

It is the time required for the response to reach the **peak value** for the first time. It is denoted by t_p . At $t=t_p$ the first derivate of the response is zero.

We know the step response of second order system for under-damped case is

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{dc(t)}{dt} = - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{dc(t)}{dt} = - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \omega_d \cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{dc(t)}{dt} = 0$ in the above equation.

$$\begin{aligned} 0 &= - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \right) [\omega_d \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta)] \\ &\Rightarrow \omega_n \sqrt{1-\delta^2} \cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta) = 0 \\ &\Rightarrow \sqrt{1-\delta^2} \cos(\omega_d t_p + \theta) - \delta \sin(\omega_d t_p + \theta) = 0 \\ &\Rightarrow \sin(\theta) \cos(\omega_d t_p + \theta) - \cos(\theta) \sin(\omega_d t_p + \theta) = 0 \\ &\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0 \\ &\Rightarrow \sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow \sin(\omega_d t_p) = 0 \\ &\Rightarrow \omega_d t_p = \pi \\ &\Rightarrow t_p = \frac{\pi}{\omega_d} \end{aligned}$$

From the above equation, we can conclude that the peak time t_p and the damped frequency ω_d are inversely proportional to each other.

Peak Overshoot

Peak overshoot M_p is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

Mathematically, we can write it as

$$M_p = c(t_p) - c(\infty)$$

Where, $c(t_p)$ is the peak value of the response, $c(\infty)$ is the final (steady state) value of the response.

At $t=t_p$, the response $c(t)$ is -

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}} \right) \sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n \left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}} \right) \sin\left(\omega_d \left(\frac{\pi}{\omega_d}\right) + \theta\right)$$
$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}} \right) (-\sin(\theta))$$

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$
$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Percentage of peak overshoot $\% M_p$ can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

From the above equation, we can conclude that the percentage of peak overshoot %Mp will decrease if the damping ratio δ increases.

Settling time

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by t_s .

The settling time for 5% tolerance band is –

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

The settling time for 2% tolerance band is –

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where, τ is the time constant and is equal to $1/\delta\omega_n$.

- ☐ Both the settling time t_s and the time constant τ are inversely proportional to the damping ratio δ .
- ☐ Both the settling time t_s and the time constant τ are independent of the system gain. That means even the system gain changes, the settling time t_s and time constant τ will never change.

Example

Let us now find the time domain specifications of a control system having the closed loop transfer function when the unit step signal is applied as an input to this control system. We know that the standard form of the transfer function of the second order closed loop control system as

$$\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

By equating these two transfer functions, we will get the un-damped natural frequency ω_n as 2 rad/sec and the damping ratio δ as 0.5.

We know the formula for damped frequency ω_d as

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

$$\omega_d = \omega_n \sqrt{1 - \delta^2}$$

Substitute, ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1 - (0.5)^2}$$

$$\Rightarrow \omega_d = 1.732 \text{ rad/sec}$$

Substitute, δ value in following relation

$$\theta = \cos^{-1} \delta$$

$$\Rightarrow \theta = \cos^{-1}(0.5) = \frac{\pi}{3} \text{ rad}$$

Substitute the above necessary values in the formula of each time domain specification and simplify in order to get the values of time domain specifications for given transfer function.

The following table shows the formulae of time domain specifications, substitution of necessary values and the final values

Time domain specification	Formula	Substitution of values in Formula	Final value
Delay time	$t_d = \frac{1+0.7\delta}{\omega_n}$	$t_d = \frac{1+0.7(0.5)}{2}$	$t_d=0.675$ sec
Rise time	$t_r = \frac{\pi-\theta}{\omega_d}$	$t_r = \frac{\pi-(\frac{\pi}{3})}{1.732}$	$t_r=1.207$ sec
Peak time	$t_p = \frac{\pi}{\omega_d}$	$t_p = \frac{\pi}{1.732}$	$t_p=1.813$ sec
% Peak overshoot	$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} \right) \times 100\%$	$\%M_p = \left(e^{-\left(\frac{0.5\pi}{\sqrt{1-(0.5)^2}}\right)} \right) \times 100\%$	$\% M_p = 16.32\%$
Settling time for 2% tolerance band	$t_s = \frac{4}{\delta\omega_n}$	$t_s = \frac{4}{(0.5)(2)}$	$t_s=4$ sec

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss} . We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} E(s)$$

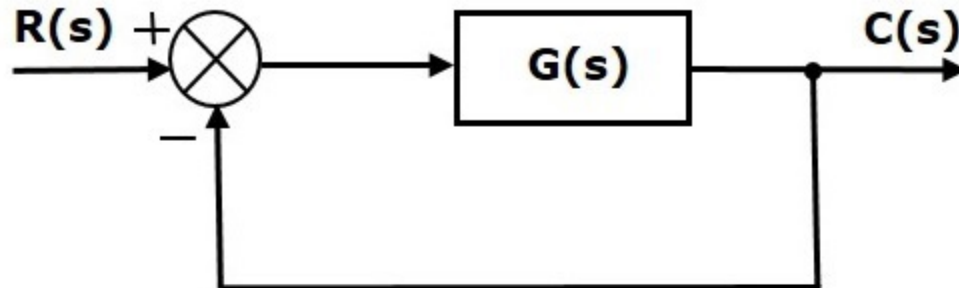
Where,

$E(s)$ is the Laplace transform of the error signal, $e(t)$

Let us discuss how to find steady state errors for unity feedback and non-unity feedback control systems one by one.

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



Where,

- $R(s)$ is the Laplace transform of the reference Input signal $r(t)$
- $C(s)$ is the Laplace transform of the output signal $c(t)$

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1 + G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1 + G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1 + G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s)}$$

The following table shows the steady state errors and the error constants for standard input signals like unit step, unit ramp & unit parabolic signals.

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s \rightarrow 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s \rightarrow 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s \rightarrow 0} s^2 G(s)$

Where, K_p , K_v and K_a are position error constant, velocity error constant and acceleration error constant respectively.

Note – If any of the above input signals has the amplitude other than unity, then multiply corresponding steady state error with that amplitude.

Note – We can't define the steady state error for the unit impulse signal because, it exists only at origin. So, we can't compare the impulse response with the unit impulse input as t denotes infinity

Example

Let us find the steady state error for an input signal $r(t) = \left(5 + 2t + \frac{t^2}{2}\right) u(t)$ of unity negative feedback control system with $G(s) = \frac{5(s+4)}{s^2(s+1)(s+20)}$

The given input signal is a combination of three signals step, ramp and parabolic. The following table shows the error constants and steady state error values for these three signals.

Input signal	Error constant	Steady state error
$r_1(t) = 5u(t)$	$K_p = \lim_{s \rightarrow 0} G(s) = \infty$	$e_{ss1} = \frac{5}{1+k_p} = 0$
$r_2(t) = 2tu(t)$	$K_v = \lim_{s \rightarrow 0} sG(s) = \infty$	$e_{ss2} = \frac{2}{K_v} = 0$
$r_3(t) = \frac{t^2}{2}u(t)$	$K_a = \lim_{s \rightarrow 0} s^2G(s) = 1$	$e_{ss3} = \frac{1}{k_a} = 1$

We will get the overall steady state error, by adding the above three steady state errors.

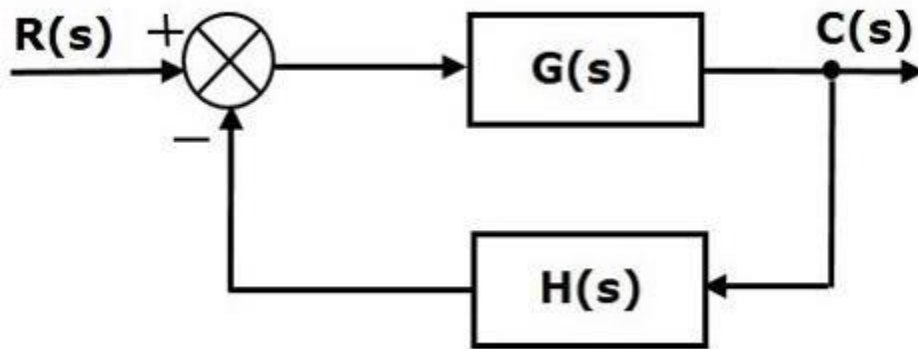
$$e_{ss} = e_{ss1} + e_{ss2} + e_{ss3}$$

$$\Rightarrow e_{ss} = 0 + 0 + 1 = 1 \Rightarrow e_{ss} = 0 + 0 + 1 = 1$$

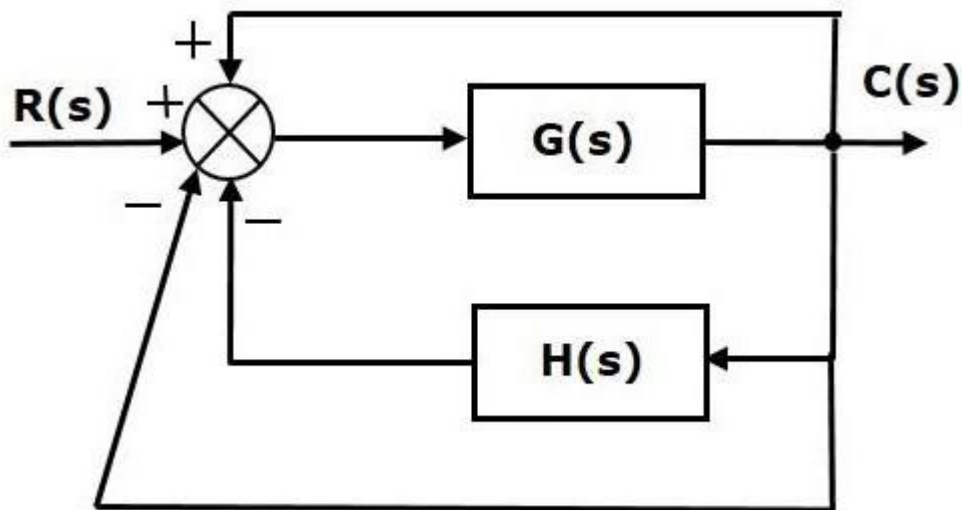
Therefore, we got the steady state error e_{ss} as **1** for this example.

Steady State Errors for Non-Unity Feedback Systems

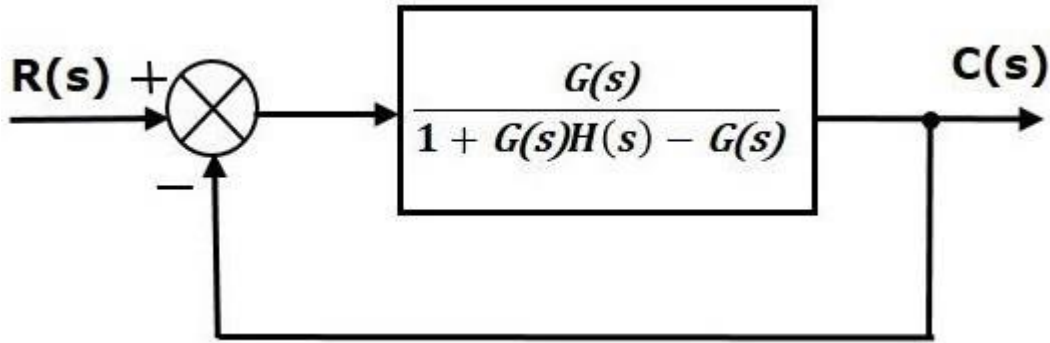
Consider the following block diagram of closed loop control system, which is having non unity negative feedback.



We can find the steady state errors only for the unity feedback systems. So, we have to convert the non-unity feedback system into unity feedback system. For this, include one unity positive feedback path and one unity negative feedback path in the above block diagram. The new block diagram looks like as shown below.



Simplify the above block diagram by keeping the unity negative feedback as it is. The following is the simplified block diagram



This block diagram resembles the block diagram of the unity negative feedback closed loop control system. Here, the single block is having the transfer function $G(s) / [1 + G(s)H(s) - G(s)]$ instead of $G(s)$. You can now calculate the steady state errors by using steady state error formula given for the unity negative feedback systems.

Note – It is meaningless to find the steady state errors for unstable closed loop systems. So, we have to calculate the steady state errors only for closed loop stable systems. This means we need to check whether the control system is stable or not before finding the steady state errors. In the next chapter, we will discuss the concepts-related stability.

The various types of controllers are used to improve the performance of control systems. In this chapter, we will discuss the basic controllers such as the proportional, the derivative and the integral controllers.

Proportional Controller

The proportional controller produces an output, which is proportional to error signal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

Therefore, the transfer function of the proportional controller is K_P .

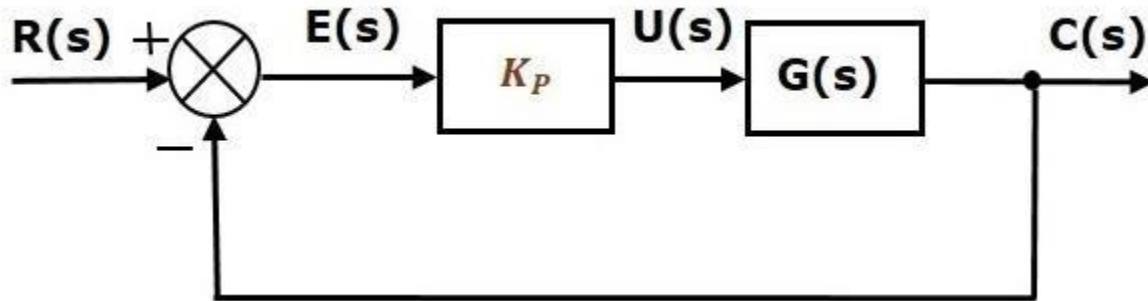
Where,

$U(s)$ is the Laplace transform of the actuating signal $u(t)$

$E(s)$ is the Laplace transform of the error signal $e(t)$

K_p is the proportionality constant

The block diagram of the unity negative feedback closed loop control system along with the proportional controller is shown in the following figure.



Derivative Controller

The derivative controller produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides.

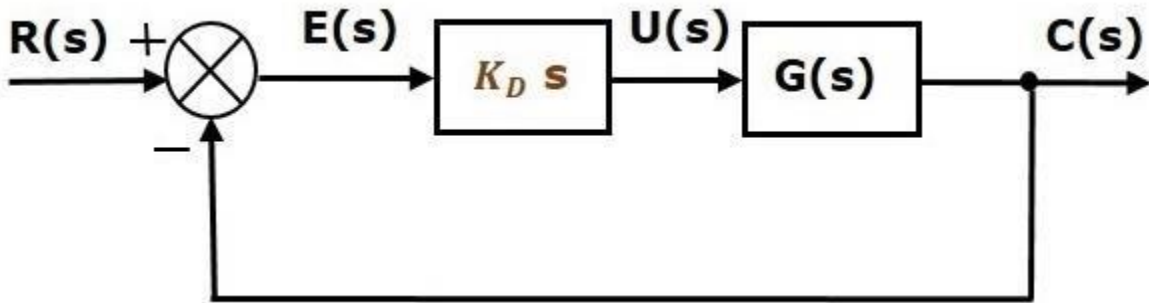
$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore, the transfer function of the derivative controller is

$K_D s$. Where, K_D is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure.



The derivative controller is used to make the unstable control system into a stable one.

Integral Controller

The integral controller produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t) dt$$

Apply Laplace transform on both the sides -

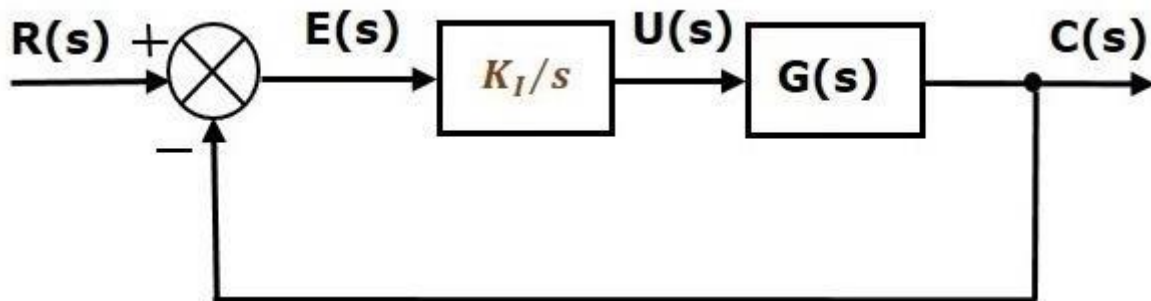
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where, K_I is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure.



The integral controller is used to decrease the steady state error. Let us now discuss about the combination of basic controllers.

Proportional Derivative (PD) Controller

The proportional derivative controller produces an output, which is the combination of the outputs of proportional and derivative controllers.

$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt}$$

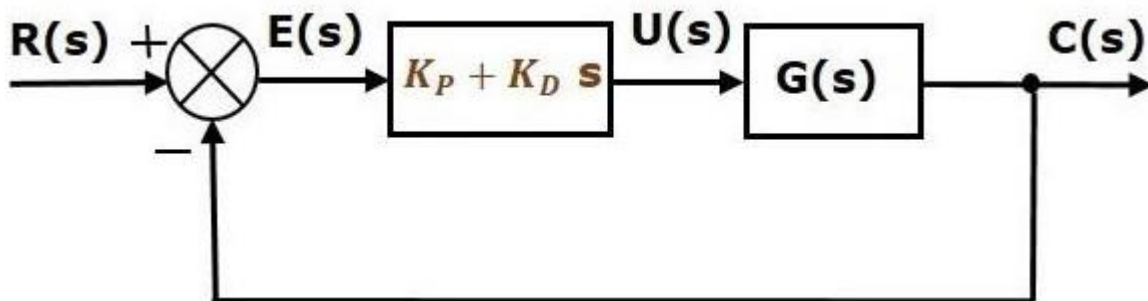
Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s)E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore, the transfer function of the proportional derivative controller is $K_P + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure.



The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

Proportional Integral (PI) Controller

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt$$

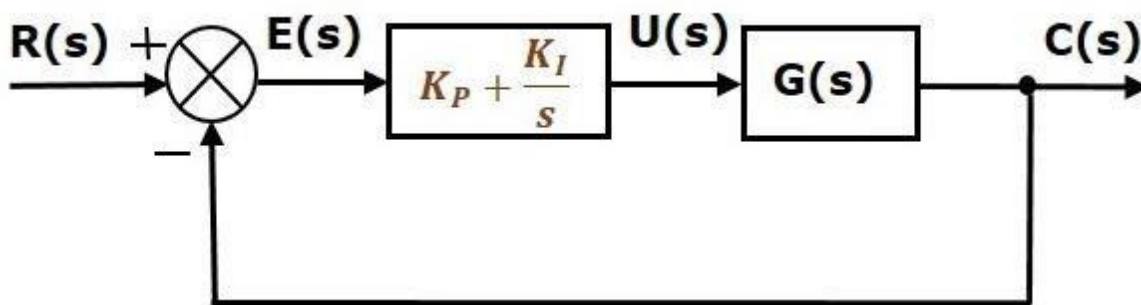
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} \right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is $K_P + \frac{K_I}{s}$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral controller is shown in the following figure.



The proportional integral controller is used to decrease the steady state error without affecting the stability of the control system.

Proportional Integral Derivative (PID) Controller

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt + K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s \right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P + \frac{K_I}{s} + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.

