Unit- II

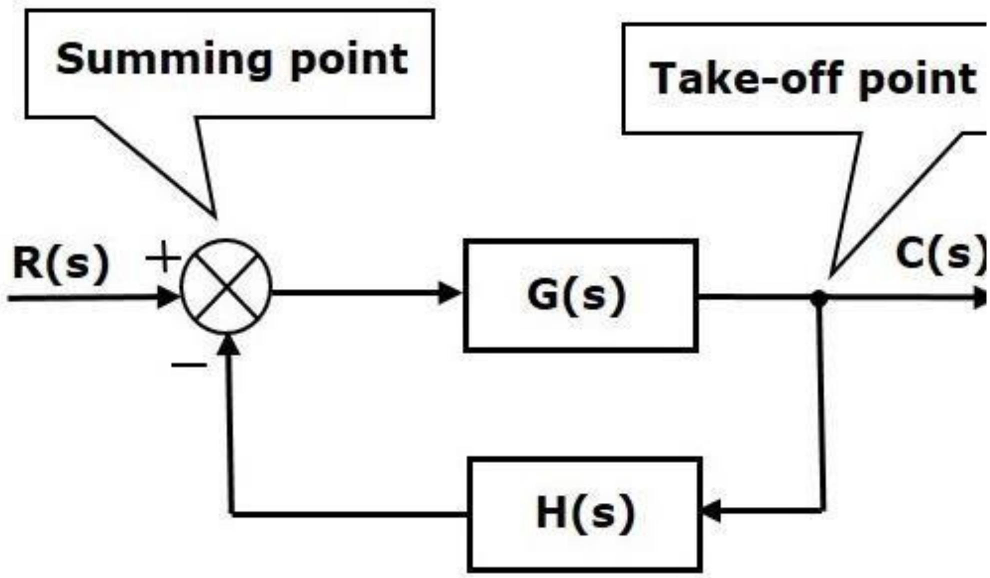
TRANSFER FUNCTION REPRESENTATION

Block Diagrams

Block diagrams consist of a single block or a combination of blocks. These are used to represent the control systems in pictorial form.

Basic Elements of Block Diagram

The basic elements of a block diagram are a block, the summing point and the take-off point. Let us consider the block diagram of a closed loop control system as shown in the following figure to identify these elements.



The above block diagram consists of two blocks having transfer functions G(s) and H(s). It is also having one summing point and one take-off point. Arrows indicate the direction of the flow of signals. Let us now discuss these elements one by one.

Block

The transfer function of a component is represented by a block. Block has single input and single output.

The following figure shows a block having input X(s), output Y(s) and the transfer function G(s).

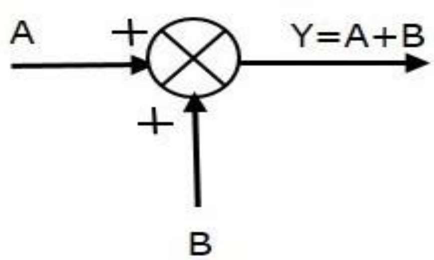
Transfer Function, $$G(s) = \frac{Y(s)}{X(s)}$$

$$\Rightarrow Y(s) = G(s)X(s)$$

Summing Point

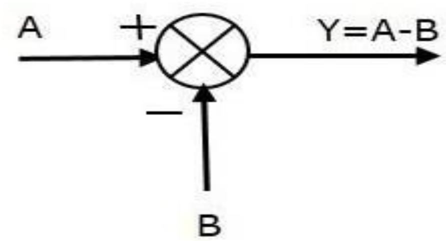
The summing point is represented with a circle having cross (X) inside it. It has two or more inputs and single output. It produces the algebraic sum of the inputs. It also performs the summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of the inputs. Let us see these three operations one by one.

The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B have a positive sign. So, the summing point produces the output, Y as **sum of A and B** i.e. = A + B.



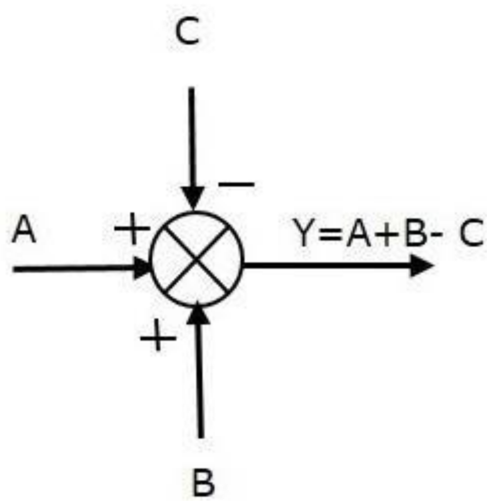
The following figure shows the summing point with two inputs (A, B) and one output (Y). Here, the inputs A and B are having opposite signs, i.e., A is having positive sign and B is having negative sign. So, the summing point produces the output **Y** as the **difference of A and B** i.e

Y = A + (-B) = A - B.



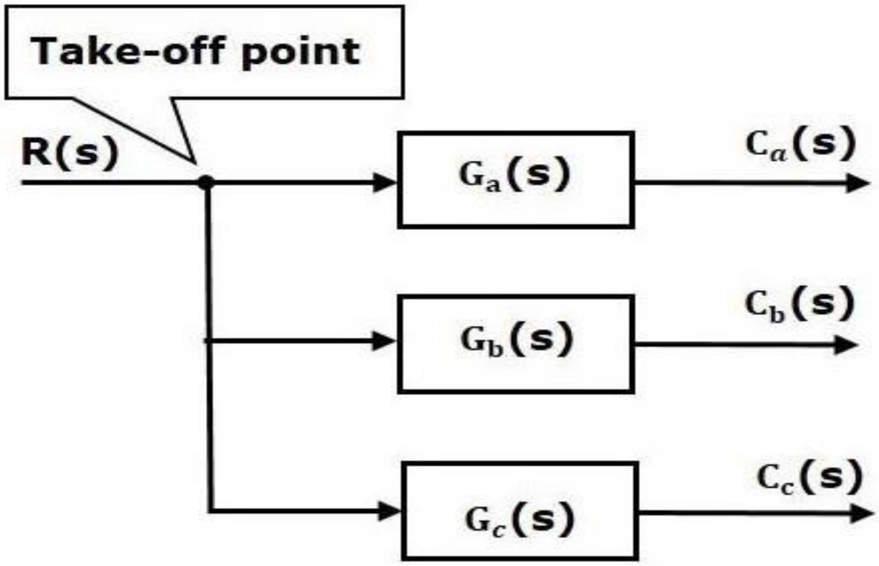
The following figure shows the summing point with three inputs (A, B, C) and one output (Y). Here, the inputs A and B are having positive signs and C is having a negative sign. So, the summing point produces the output **Y** as

Y = A + B + (−C) = A + B − C.

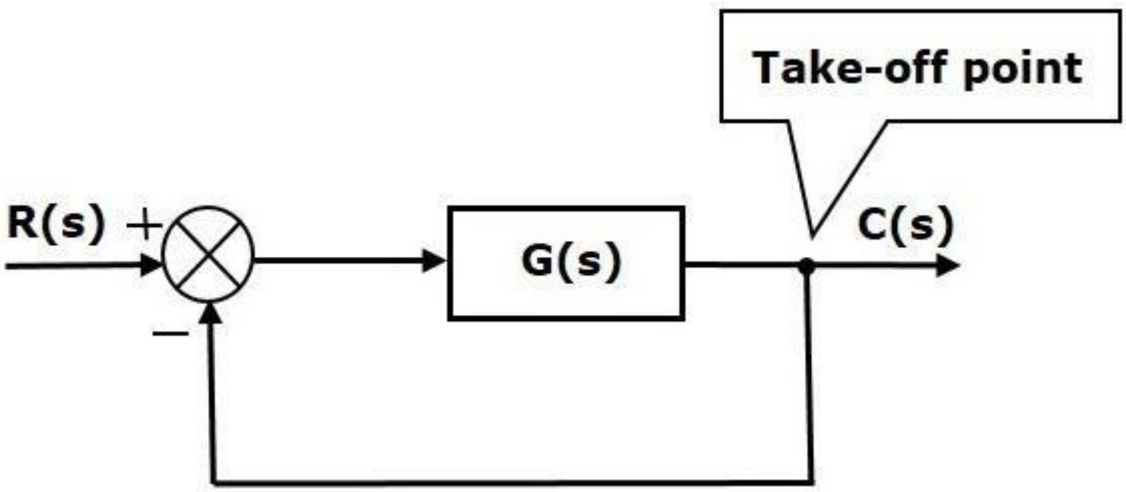


Take-off Point

The take-off point is a point from which the same input signal can be passed through more than one branch. That means with the help of take-off point, we can apply the same input to one or more blocks, summing points. In the following figure, the take-off point is used to connect the same input, R(s) to two more blocks.



In the following figure, the take-off point is used to connect the output C(s), as one of the inputs to the summing point.



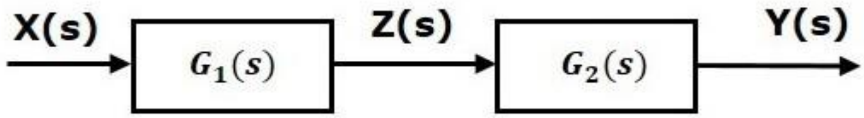
Block diagram algebra is nothing but the algebra involved with the basic elements of the block diagram. This algebra deals with the pictorial representation of algebraic equations.

Basic Connections for Blocks

There are three basic types of connections between two blocks.

Series Connection

Series connection is also called **cascade connection**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in series.



For this combination, we will get the output $Y(s)$ as

$$Y(s) = G_2(s)Z(s)$$

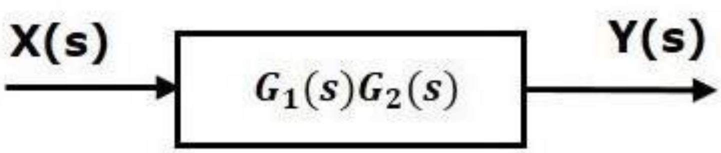
Where, $Z(s) = G_1(s)X(s)$

$$\Rightarrow Y(s) = G_2(s)[G_1(s)X(s)] = G_1(s)G_2(s)X(s)$$

$$\Rightarrow Y(s) = \{G_1(s)G_2(s)\}X(s)$$

Compare this equation with the standard form of the output equation, $Y(s) = G(s)X(s)$. Where, $G(s) = G_1(s)G_2(s)$.

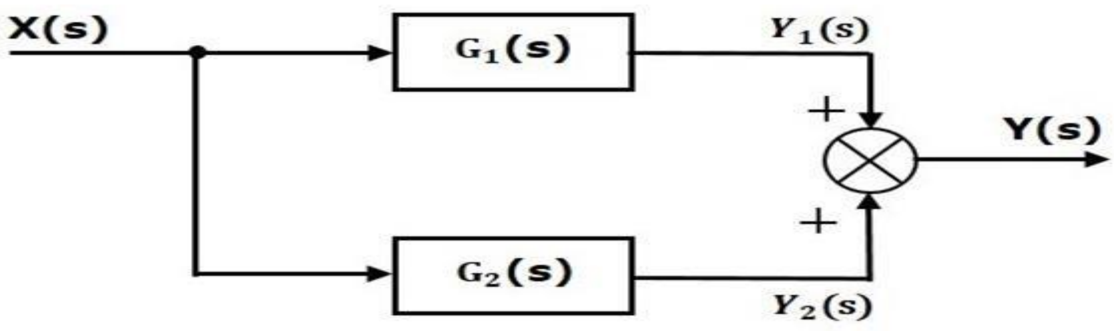
That means we can represent the **series connection** of two blocks with a single block. The transfer function of this single block is the **product of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent series connection of 'n' blocks with a single block. The transfer function of this single block is the product of the transfer functions of all those 'n' blocks.

Parallel Connection

The blocks which are connected in **parallel** will have the **same input**. In the following figure, two blocks having transfer functions G1(s)G1(s) and G2(s)G2(s) are connected in parallel. The outputs of these two blocks are connected to the summing point.



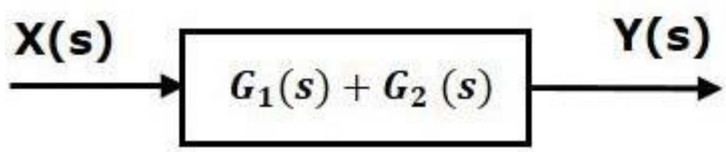
$$Y(s) = Y_1(s) + Y_2(s)$$

$$Y_1(s) = G_1(s)X(s) \text{ and } Y_2(s) = G_2(s)X(s)$$

$$\Rightarrow Y(s) = G_1(s)X(s) + G_2(s)X(s) = \{G_1(s) + G_2(s)\}X(s)$$

$$G(s) = G_1(s) + G_2(s).$$

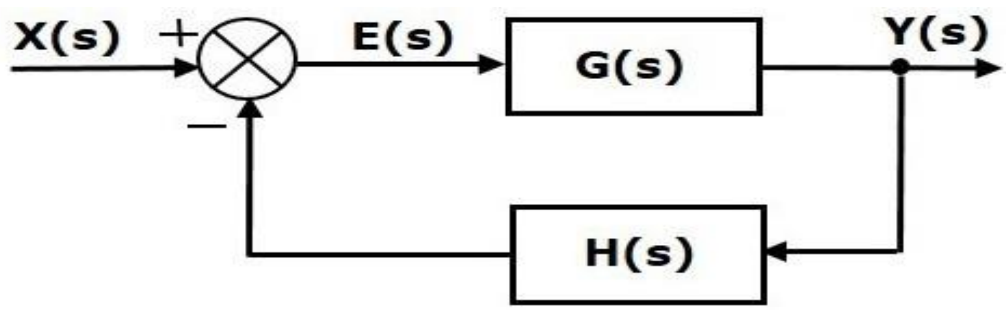
That means we can represent the **parallel connection** of two blocks with a single block. The transfer function of this single block is the **sum of the transfer functions** of those two blocks. The equivalent block diagram is shown below.



Similarly, you can represent parallel connection of 'n' blocks with a single block. The transfer function of this single block is the algebraic sum of the transfer functions of all those 'n' blocks.

Feedback Connection

As we discussed in previous chapters, there are two types of **feedback** — positive feedback and negative feedback. The following figure shows negative feedback control system. Here, two blocks having transfer functions G(s)G(s) and H(s)H(s) form a closed loop.



The output of the summing point is -

$$E(s) = X(s) - H(s)Y(s)$$

The output $Y(s)$ is -

$$Y(s) = E(s)G(s)$$

Substitute $E(s)$ value in the above equation.

$$Y(s) = \{X(s) - H(s)Y(s)\}G(s)\}$$

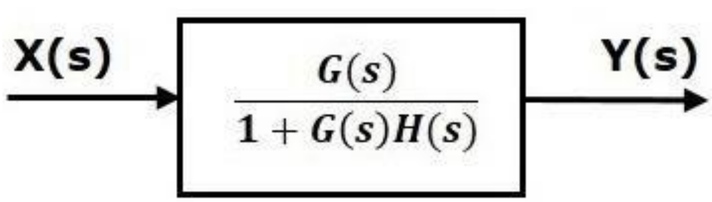
$$Y(s)\{1 + G(s)H(s)\} = X(s)G(s)\}$$

$$\Rightarrow \frac{Y(s)}{X(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Therefore, the negative feedback closed loop transfer function is :

$$\frac{G(s)}{1+G(s)H(s)}$$

This means we can represent the negative feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the negative feedback. The equivalent block diagram is shown below.



Similarly, you can represent the positive feedback connection of two blocks with a single block. The transfer function of this single block is the closed loop transfer function of the positive feedback, i.e.,

$$\frac{G(s)}{1-G(s)H(s)}$$

Block Diagram Algebra for Summing Points

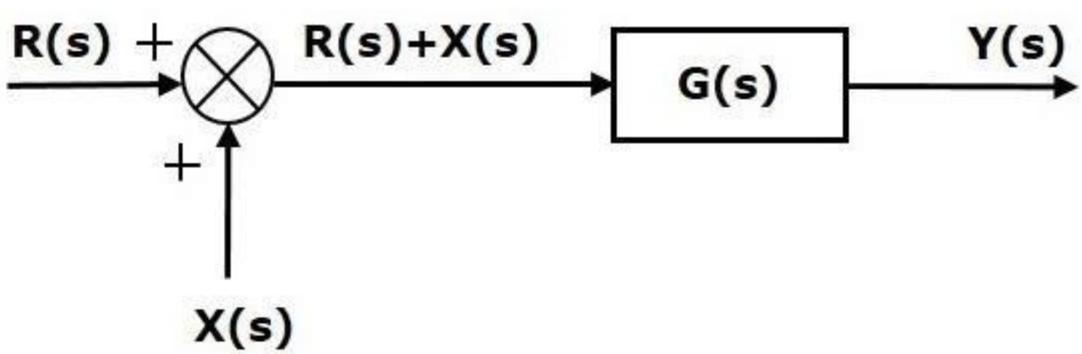
There are two possibilities of shifting summing points with respect to blocks –

- Shifting summing point after the block
- Shifting summing point before the block

Let us now see what kind of arrangements need to be done in the above two cases one by one.

Shifting the Summing Point before a Block to after a Block

Consider the block diagram shown in the following figure. Here, the summing point is present before the block.



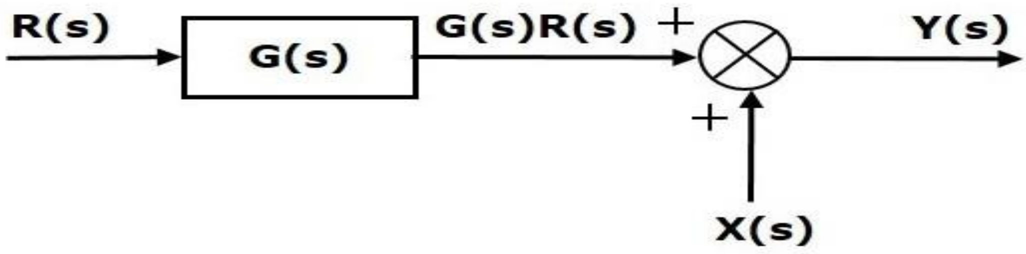
Summing point has two inputs $R(s)$ and $X(s)$

The output of Summing point is $\{R(s) + X(s)\}$.

So, the input to the block $G(s)$ is $\{R(s) + X(s)\}$ and the output of it is –

$$Y(s) = G(s)\{R(s) + X(s)\}$$

$$\Rightarrow Y(s) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 1)}$$



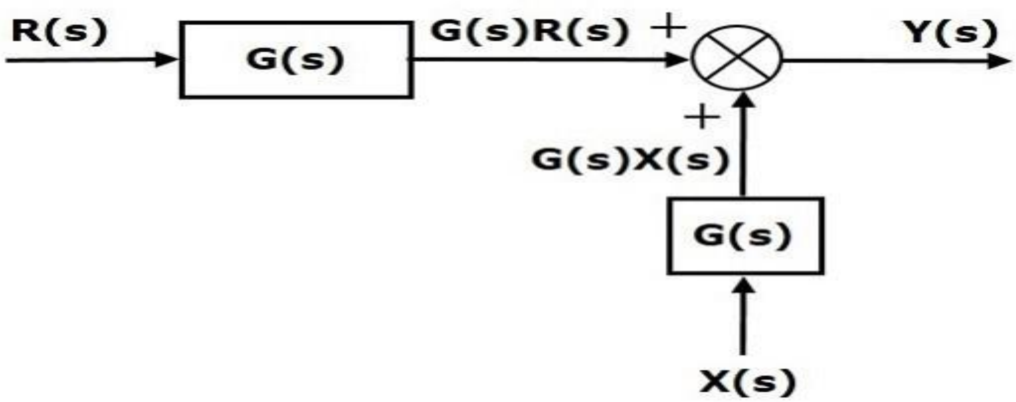
Output of the block $G(s)$ is $G(s)R(s)$.

The output of the summing point is

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 2)}$$

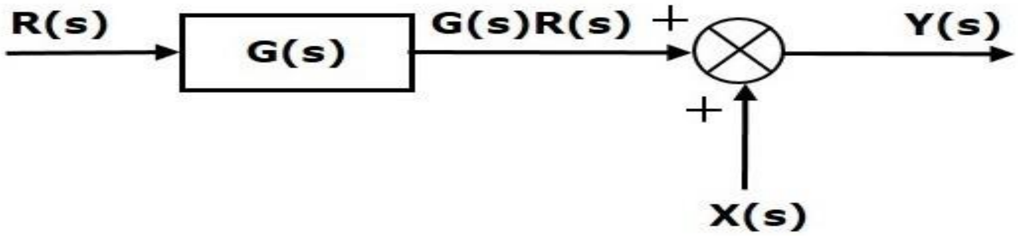
Compare Equation 1 and Equation 2.

The first term ‘G(s)R(s)′‘G(s)R(s)′ is same in both the equations. But, there is difference in the second term. In order to get the second term also same, we require one more block G(s)G(s). It is having the input X(s)X(s) and the output of this block is given as input to summing point instead of X(s)X(s). This block diagram is shown in the following figure.



Shifting Summing Point Before the Block

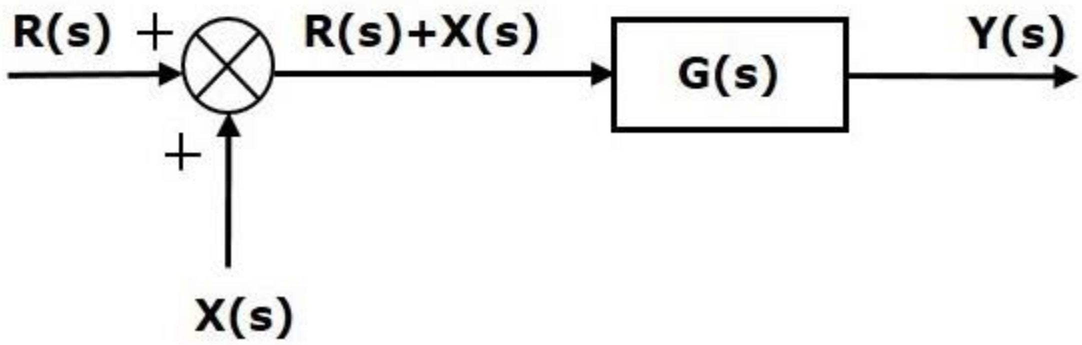
Consider the block diagram shown in the following figure. Here, the summing point is present after the block.



Output of this block diagram is -

$$Y(s) = G(s)R(s) + X(s) \qquad \text{(Equation 3)}$$

Now, shift the summing point before the block. This block diagram is shown in the following figure.

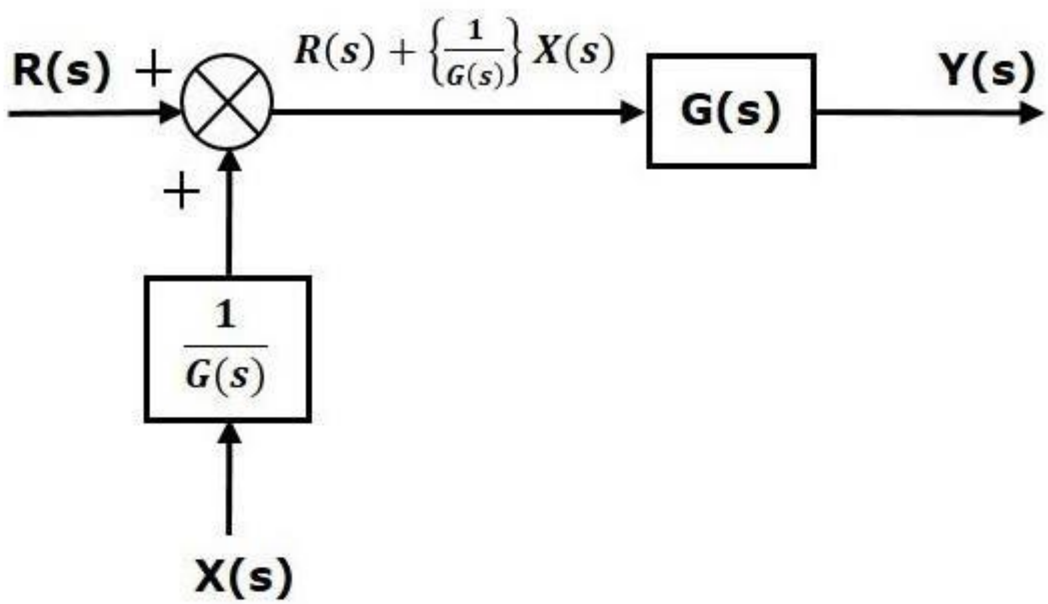


Output of this block diagram is -

$$Y(S) = G(s)R(s) + G(s)X(s) \qquad \text{(Equation 4)}$$

Compare Equation 3 and Equation 4,

The first term 'G(s)R(s)' is same in both equations. But, there is difference in the second term. In order to get the second term also same, we require one more block 1/G(s). It is having the input X(s) and the output of this block is given as input to summing point instead of X(s). This block diagram is shown in the following figure.



Block Diagram Algebra for Take-off Points

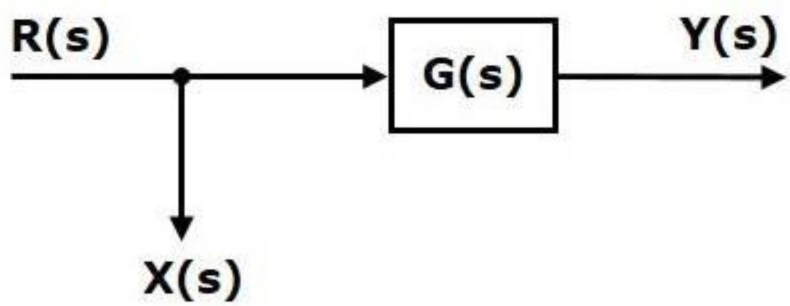
There are two possibilities of shifting the take-off points with respect to blocks –

- Shifting take-off point after the block
- Shifting take-off point before the block

Let us now see what kind of arrangements is to be done in the above two cases, one by one.

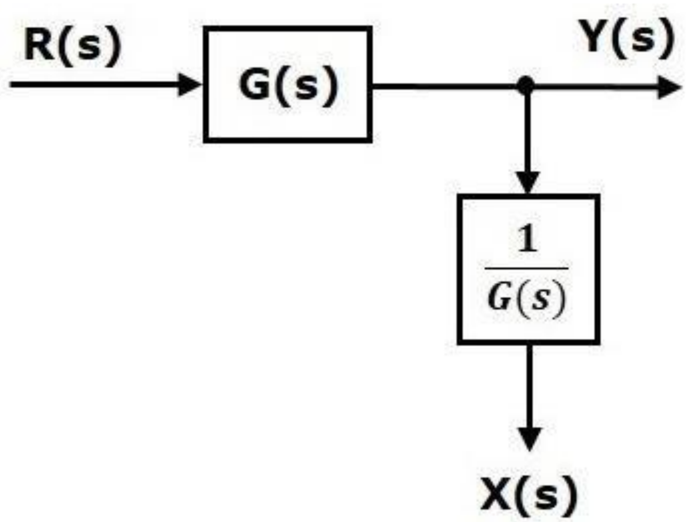
Shifting a Take-off Point form a Position before a Block to a position after the Block

Consider the block diagram shown in the following figure. In this case, the take-off point is present before the block.



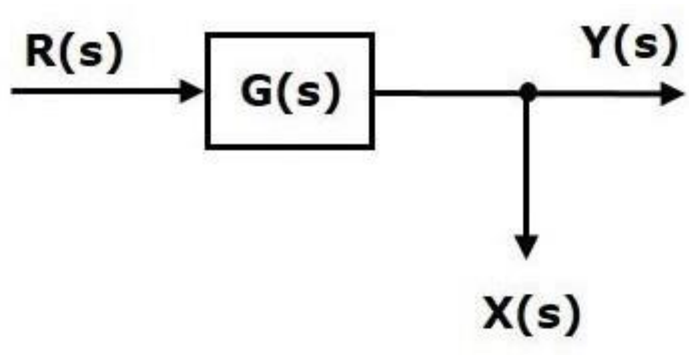
Here, $X(s) = R(s)$ and $Y(s) = G(s)R(s)$

When you shift the take-off point after the block, the output Y(s) will be same. But, there is difference in X(s) value. So, in order to get the same X(s) value, we require one more block 1/G(s). It is having the input Y(s) and the output is X(s) this block diagram is shown in the following figure.



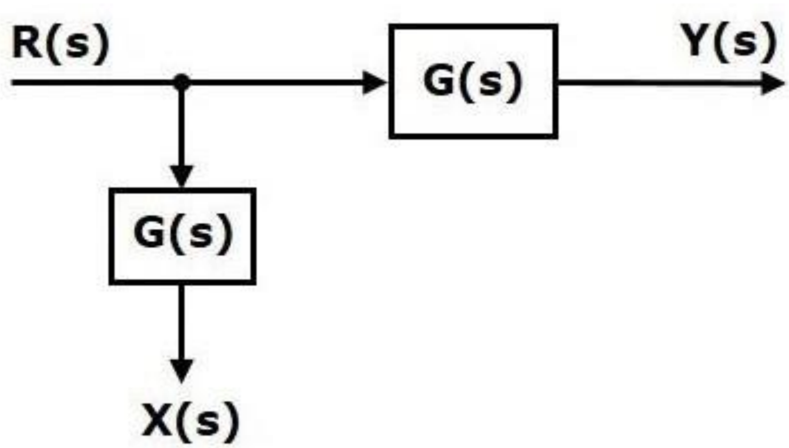
Shifting Take-off Point from a Position after a Block to a position before the Block

Consider the block diagram shown in the following figure. Here, the take-off point is present after the block.



Here, $X(s) = Y(s) = G(s)R(s)$

When you shift the take-off point before the block, the output Y(s) will be same. But, there is difference in X(s) value. So, in order to get same X(s) value, we require one more block G(s) It is having the input R(s) and the output is X(s). This block diagram is shown in the following figure.



The concepts discussed in the previous chapter are helpful for reducing (simplifying) the block diagrams.

Block Diagram Reduction Rules

Follow these rules for simplifying (reducing) the block diagram, which is having many blocks, summing points and take-off points.

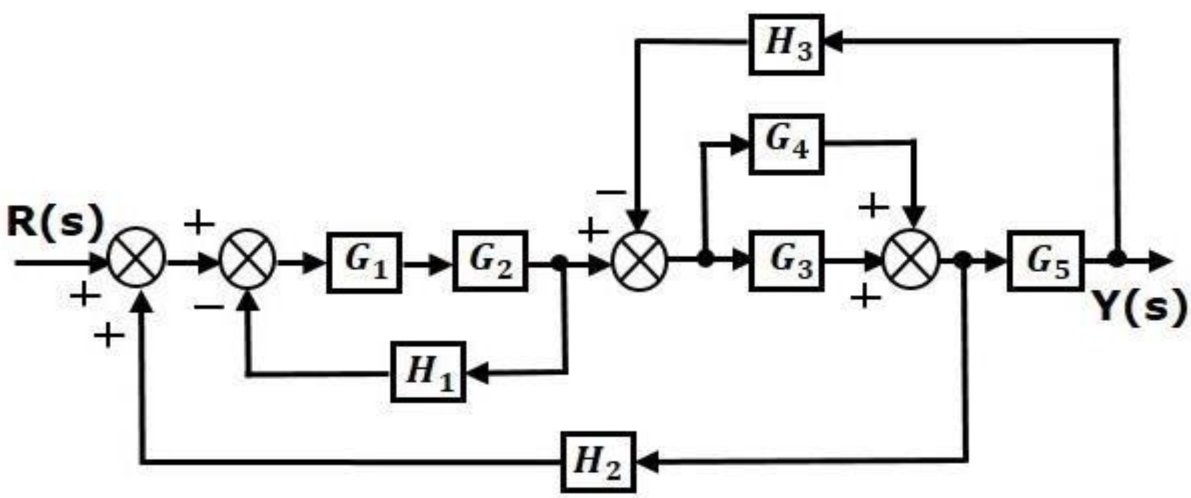
- **Rule 1 −** Check for the blocks connected in series and simplify.
- **Rule 2 −** Check for the blocks connected in parallel and simplify.

- **Rule 3 –** Check for the blocks connected in feedback loop and simplify.
- **Rule 4 –** If there is difficulty with take-off point while simplifying, shift it towards right.
- **Rule 5 –** If there is difficulty with summing point while simplifying, shift it towards left.
- **Rule 6 –** Repeat the above steps till you get the simplified form, i.e., single block.

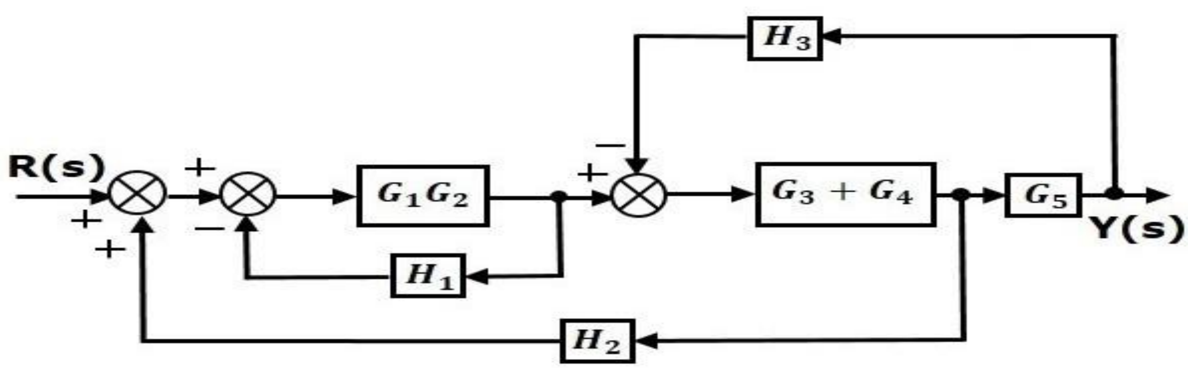
Note – The transfer function present in this single block is the transfer function of the overall block diagram.

Example

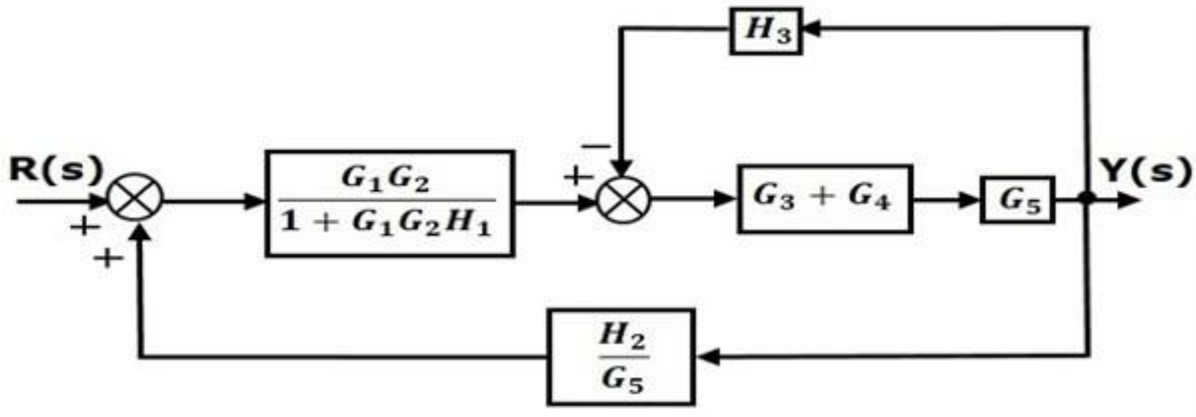
Consider the block diagram shown in the following figure. Let us simplify (reduce) this block diagram using the block diagram reduction rules.



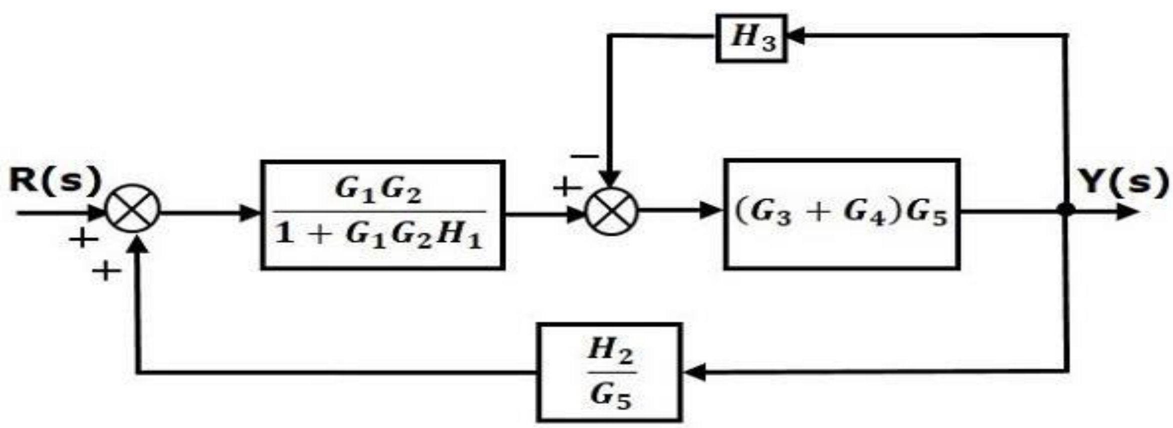
Step 1 – Use Rule 1 for blocks G_1 and G_2. Use Rule 2 for blocks G_3 and G_4. The modified block diagram is shown in the following figure.



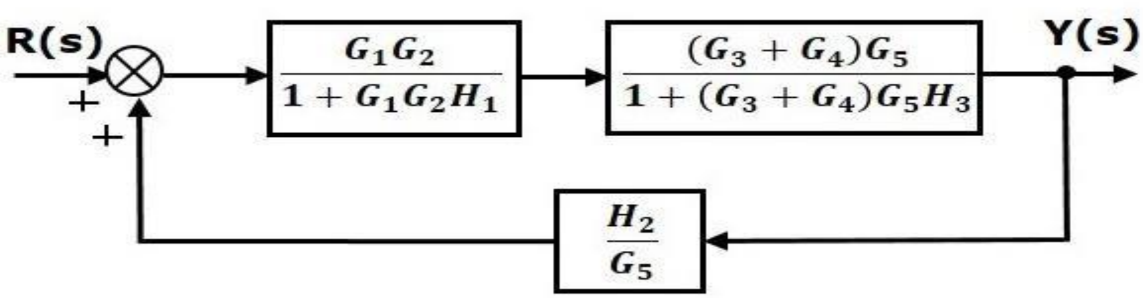
Step 2 – Use Rule 3 for blocks G_1G_2 and H_1. Use Rule 4 for shifting take-off point after the block G_5. The modified block diagram is shown in the following figure.



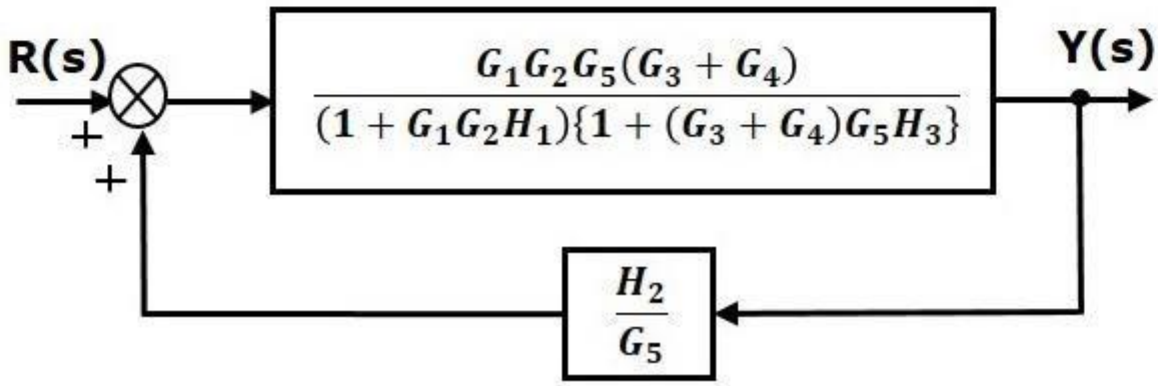
Step 3 – Use Rule 1 for blocks (G_3+G_4) and G_5. The modified block diagram is shown in the following figure.



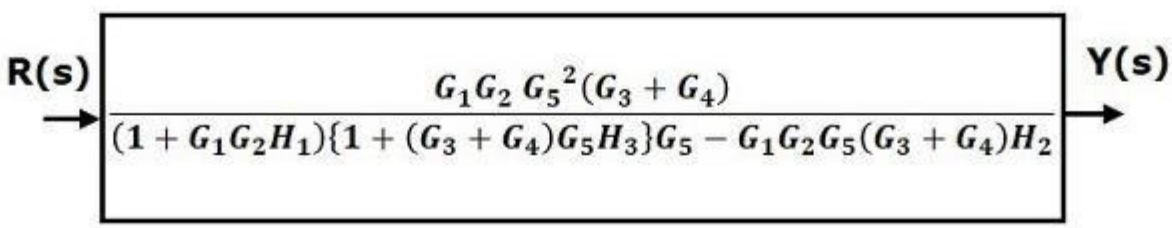
Step 4 – Use Rule 3 for blocks $(G_3+G_4)G_5$ and H_3. The modified block diagram is shown in the following figure.



Step 5 – Use Rule 1 for blocks connected in series. The modified block diagram is shown in the following figure.



Step 6 – Use Rule 3 for blocks connected in feedback loop. The modified block diagram is shown in the following figure. This is the simplified block diagram.



Therefore, the transfer function of the system is

$$\frac{Y(s)}{R(s)}=\frac{G_1G_2G_5^2(G_3+G_4)}{(1+G_1G_2H_1)\{1+(G_3+G_4)G_5H_3\}G_5-G_1G_2G_5(G_3+G_4)H_2}$$

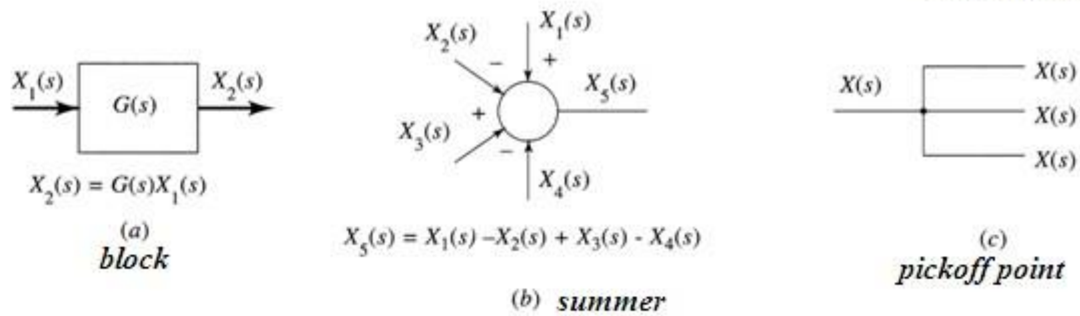
Note – Follow these steps in order to calculate the transfer function of the block diagram having multiple inputs.

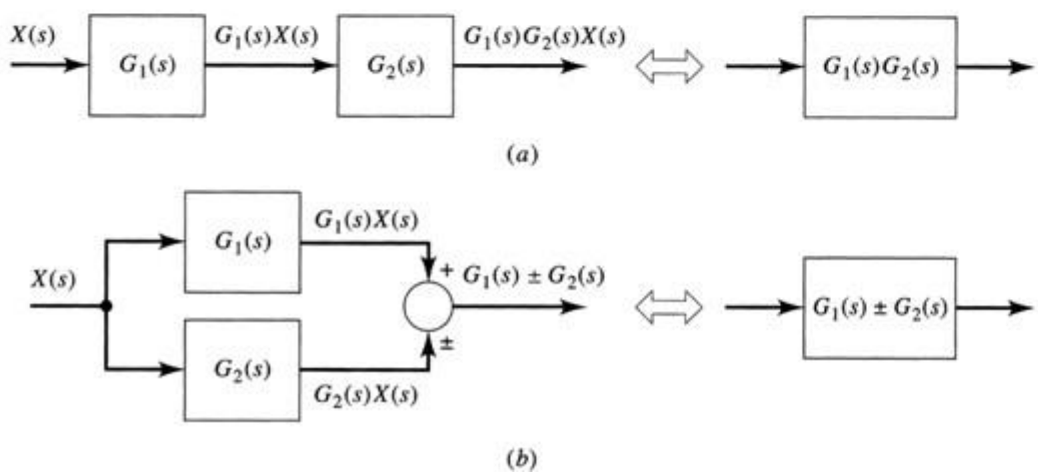
- **Step 1** – Find the transfer function of block diagram by considering one input at a time and make the remaining inputs as zero.
- **Step 2** – Repeat step 1 for remaining inputs.
- **Step 3** – Get the overall transfer function by adding all those transfer functions.

The block diagram reduction process takes more time for complicated systems because; we have to draw the (partially simplified) block diagram after each step. So, to overcome this drawback, use signal flow graphs (representation).

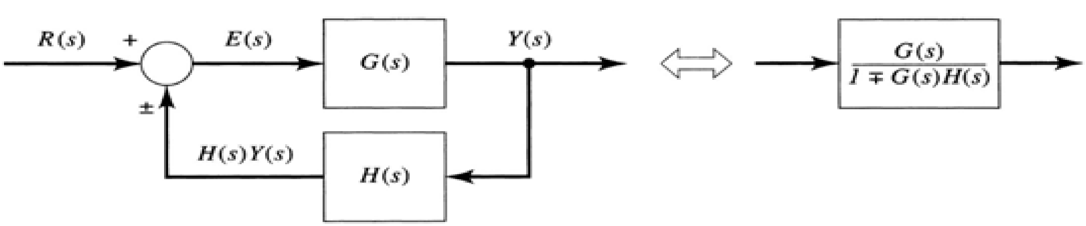
Block Diagram Reduction- Summary

Automatic control





Automatic control

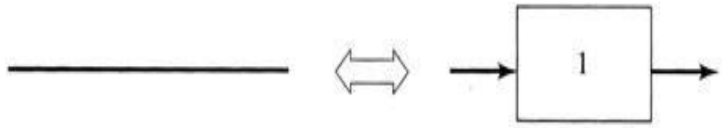


$$Y(s) = G(s)E(s)$$

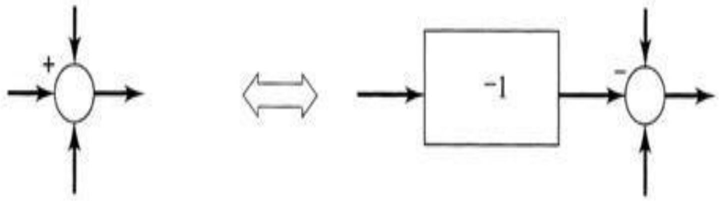
$$E(s) = R(s) \pm H(s)Y(s)$$

$$Y(s) = G(s)[R(s) \pm H(s)Y(s)] = G(s)R(s) \pm G(s)H(s)Y(s)$$

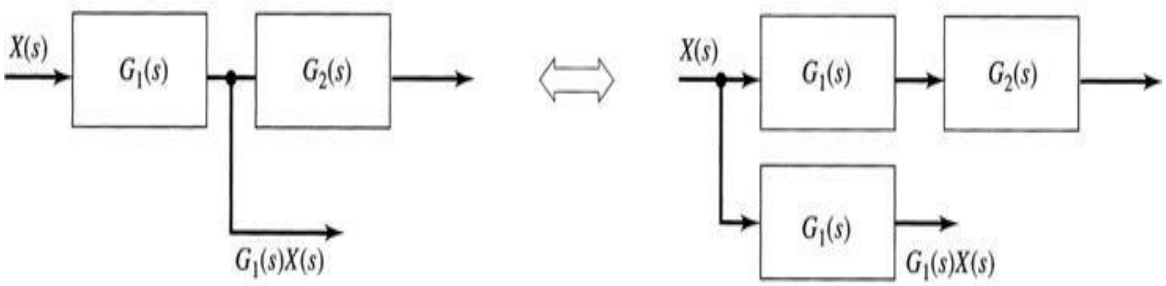
$$T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 \mp G(s)H(s)}$$



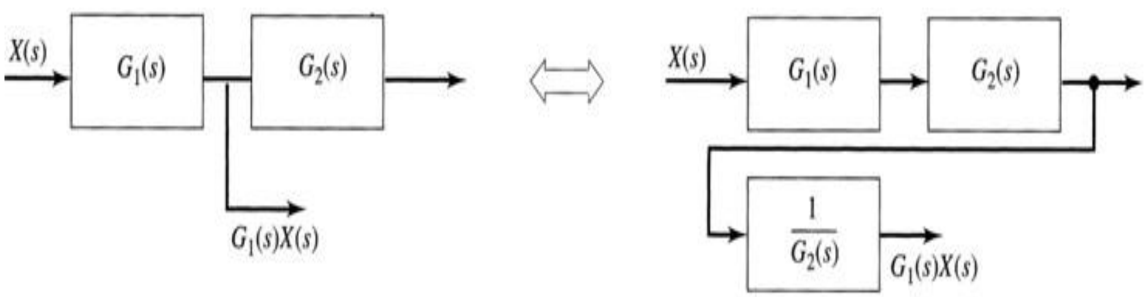
(*a*) Insertion or removal of unity gain



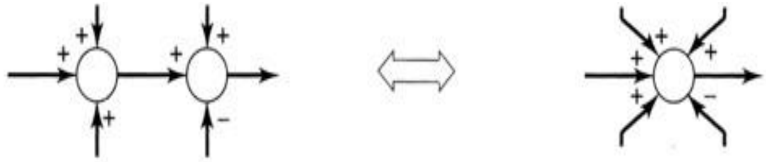
(*b*) Changing a summer sign



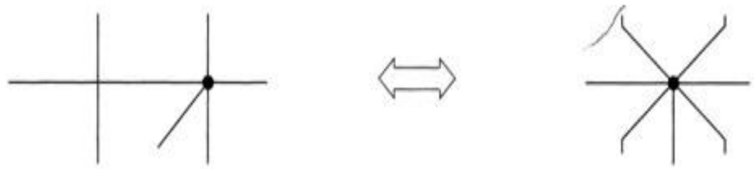
(*c*) Moving a pickoff point back



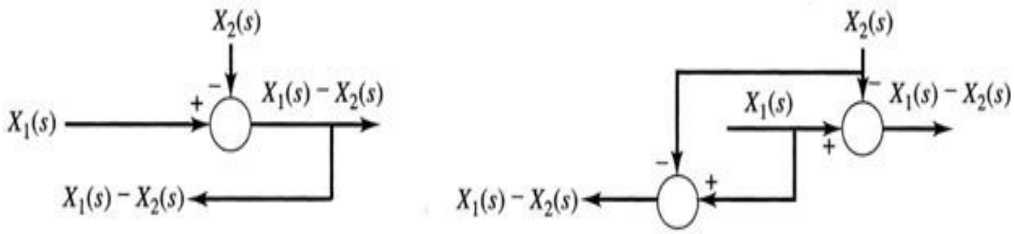
(*d*) Moving a pickoff point forward



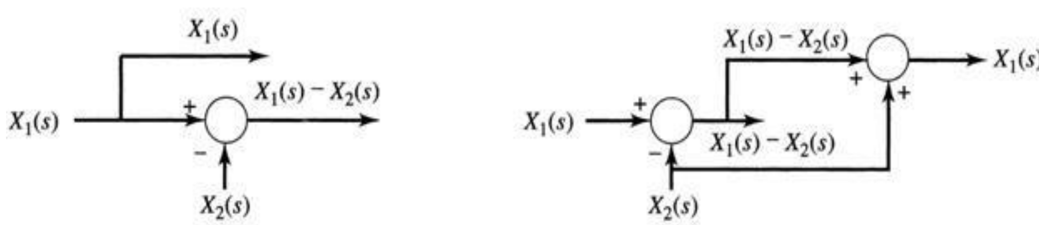
(*e*) Combining or expanding summations



(*f*) Combining or expanding junctions

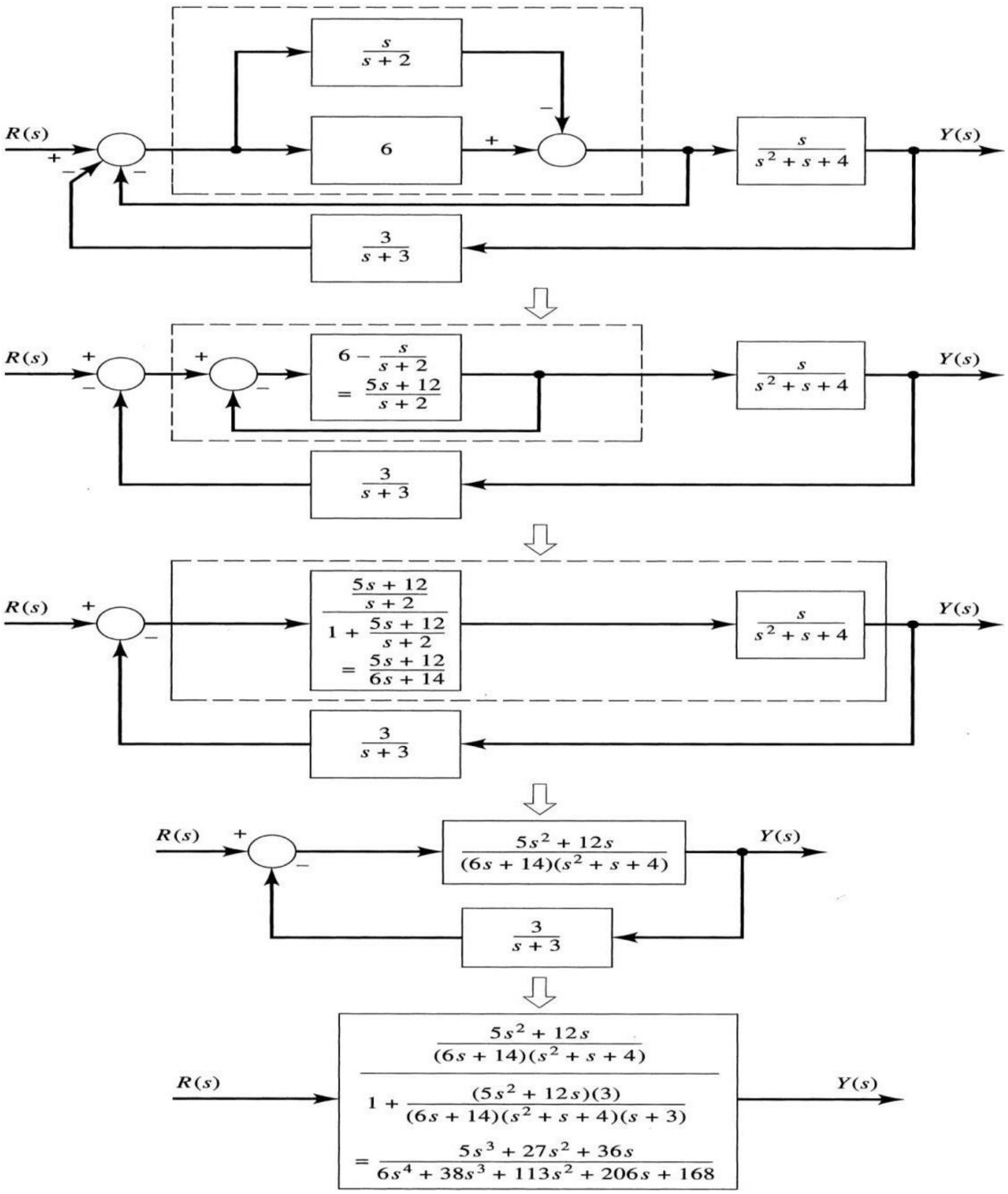


(*g*) Moving a pickoff point behind a summation



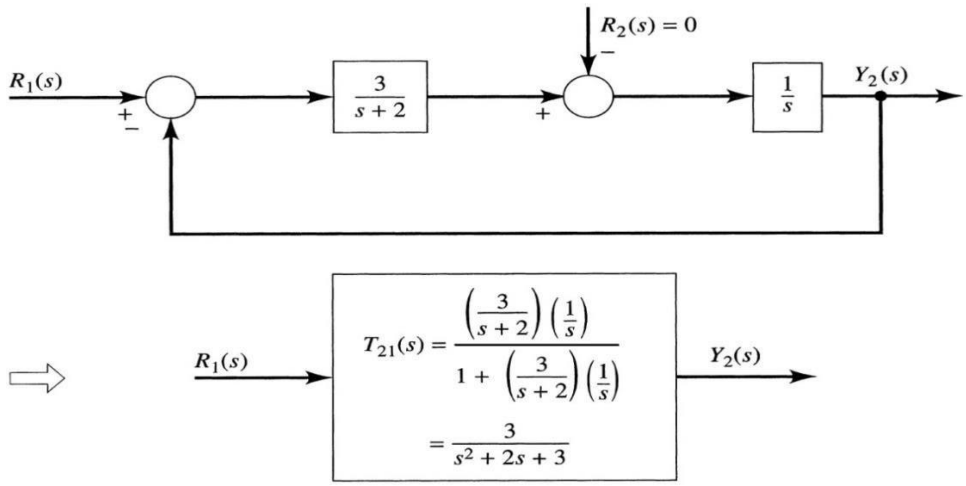
(*h*) Moving a pickoff point forward of a summation

Example-1:

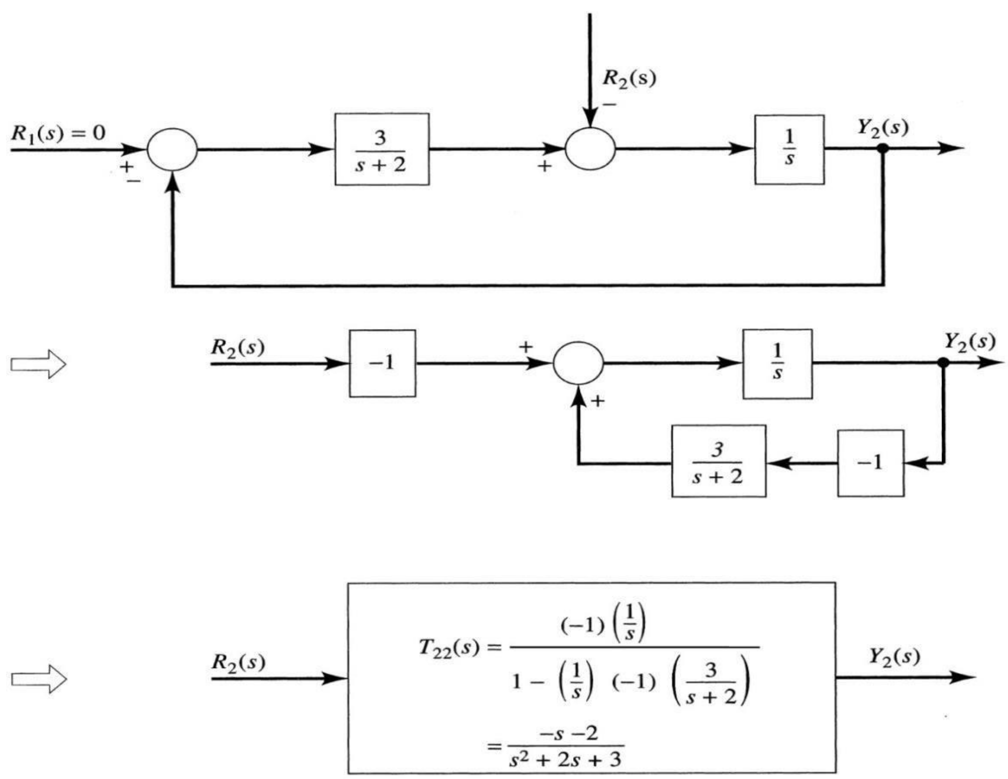


$R(s)$
$Y(s)$
$\frac{s}{s+2}$
6
$\frac{s}{s^2+s+4}$
$\frac{3}{s+3}$
$6 - \frac{s}{s+2} = \frac{5s+12}{s+2}$
$\frac{\frac{5s+12}{s+2}}{1+\frac{5s+12}{s+2}} = \frac{5s+12}{6s+14}$
$\frac{5s^2+12s}{(6s+14)(s^2+s+4)}$
$\frac{\frac{5s^2+12s}{(6s+14)(s^2+s+4)}}{1+\frac{(5s^2+12s)(3)}{(6s+14)(s^2+s+4)(s+3)}} = \frac{5s^3+27s^2+36s}{6s^4+38s^3+113s^2+206s+168}$

Example-2:


$R_2(s) = 0$
$-$
$R_1(s)$
$+$
$-$
$\frac{3}{s+2}$
$+$
$\frac{1}{s}$
$Y_2(s)$
$R_1(s)$
$T_{21}(s) = \dfrac{\left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)}{1 + \left(\frac{3}{s+2}\right)\left(\frac{1}{s}\right)} = \dfrac{3}{s^2+2s+3}$
$Y_2(s)$

(a)


$R_2(s)$
$-$
$R_1(s) = 0$
$+$
$-$
$\frac{3}{s+2}$
$+$
$\frac{1}{s}$
$Y_2(s)$
$R_2(s)$
-1
$+$
$\frac{1}{s}$
$Y_2(s)$
$+$
$\frac{3}{s+2}$
-1
$R_2(s)$
$T_{22}(s) = \dfrac{(-1)\left(\frac{1}{s}\right)}{1 - \left(\frac{1}{s}\right)(-1)\left(\frac{3}{s+2}\right)} = \dfrac{-s-2}{s^2+2s+3}$
$Y_2(s)$

(b)

Signal flow graph is a graphical representation of algebraic equations. In this chapter, let us discuss the basic concepts related signal flow graph and also learn how to draw signal flow graphs.

Basic Elements of Signal Flow Graph

Nodes and branches are the basic elements of signal flow

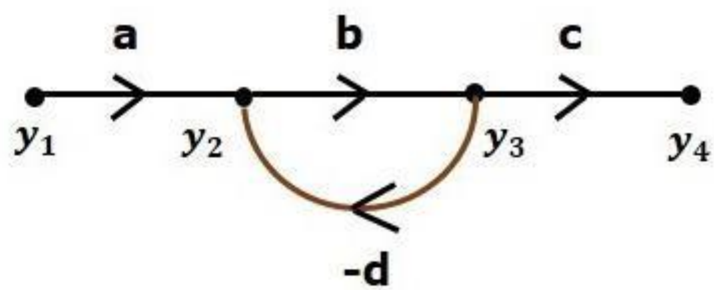
graph. Node

Node is a point which represents either a variable or a signal. There are three types of nodes — input node, output node and mixed node.

- **Input Node** − It is a node, which has only outgoing branches.
- **Output Node** − It is a node, which has only incoming branches.
- **Mixed Node** − It is a node, which has both incoming and outgoing branches.

Example

Let us consider the following signal flow graph to identify these nodes.



- The **nodes** present in this signal flow graph are y_1, y_2, y_3 and y_4.
- y_1 and y_4 are the **input node** and **output node** respectively.
- y_2 and y_3 are **mixed nodes**.

Branch

Branch is a line segment which joins two nodes. It has both **gain** and **direction**. For example, there are four branches in the above signal flow graph. These branches have **gains** of **a, b, c** and **-d**.

Construction of Signal Flow Graph

Let us construct a signal flow graph by considering the following algebraic equations –

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

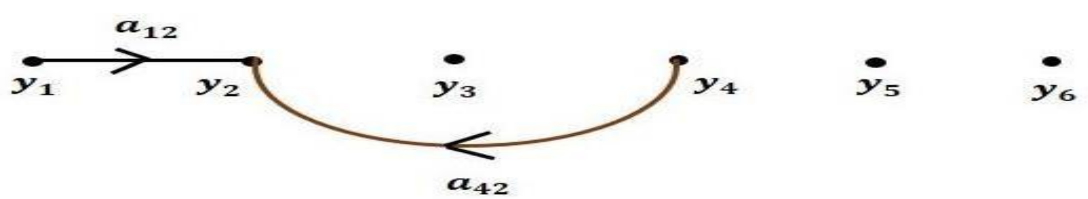
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53} and a_{35}.

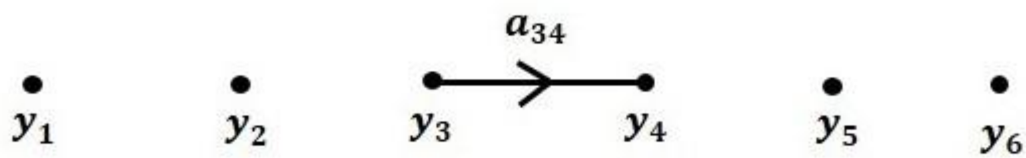
To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below –

Step 1 – Signal flow graph for $y_2 = a_{13}y_1 + a_{42}y_4$ is shown in the following figure.

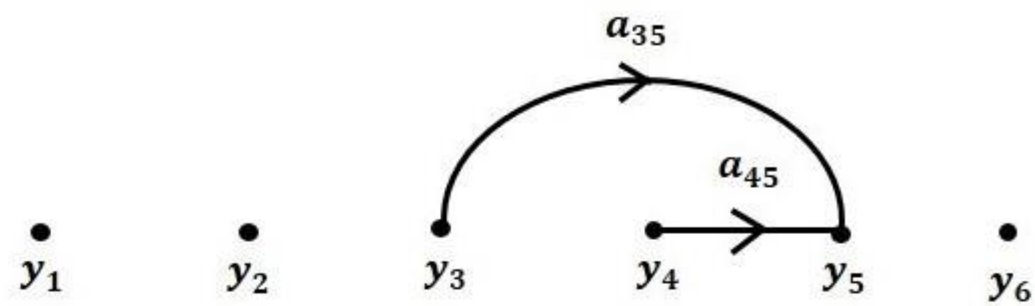


Step 2 – Signal flow graph for $y_3 = a_{23}y_2 + a_{53}y_5$ is shown in the following figure.

Step 3 – Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.

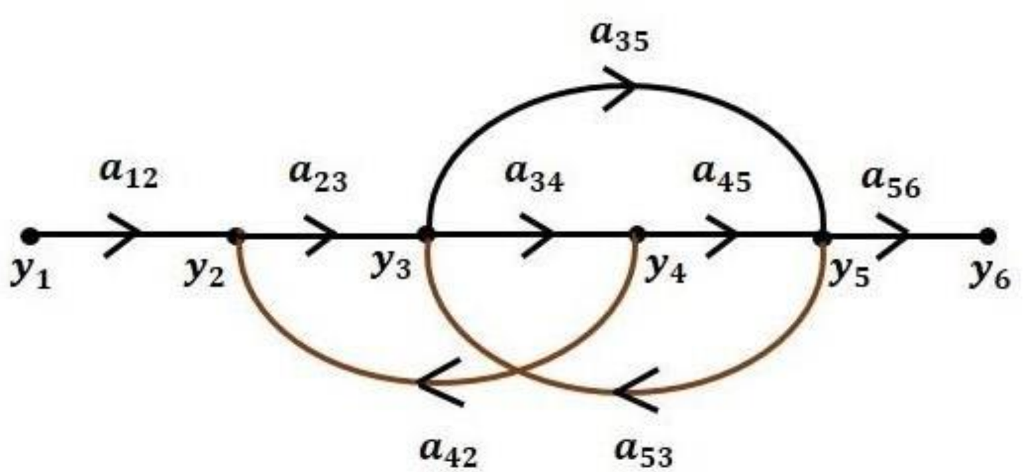


Step 4 – Signal flow graph for $y_5 = a_{45}y_4 + a_{35}y_3$ is shown in the following figure.



Step 5 – Signal flow graph for $y_6 = a_{56}y_5$ is shown in the following figure.

Step 6 – Signal flow graph of overall system is shown in the following figure.



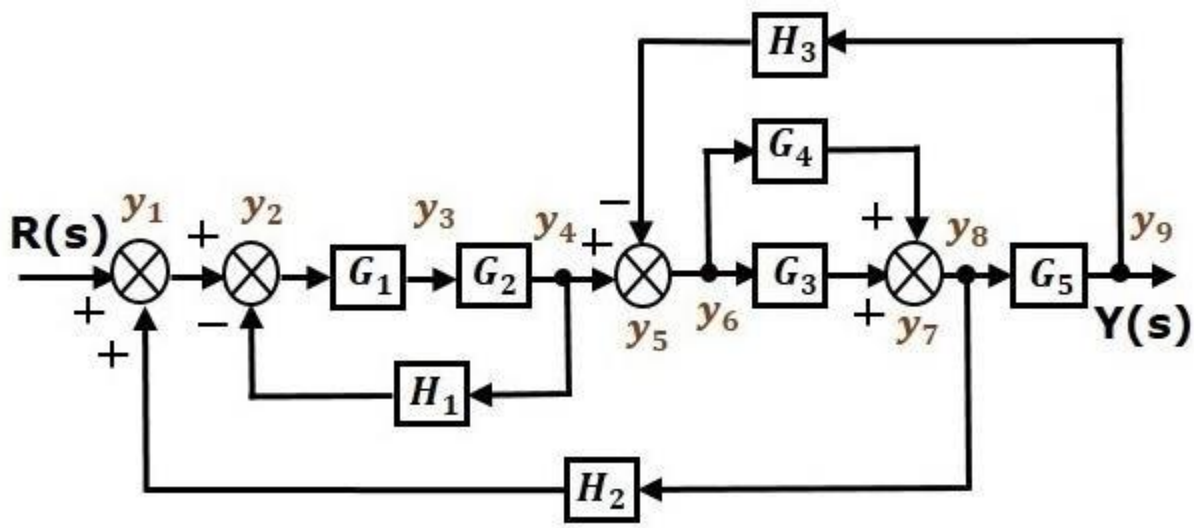
Conversion of Block Diagrams into Signal Flow Graphs

Follow these steps for converting a block diagram into its equivalent signal flow graph.

- Represent all the signals, variables, summing points and take-off points of block diagram as **nodes** in signal flow graph.
- Represent the blocks of block diagram as **branches** in signal flow graph.
- Represent the transfer functions inside the blocks of block diagram as**gains** of the branches in signal flow graph.
- Connect the nodes as per the block diagram. If there is connection between two nodes (but there is no block in between), then represent the gain of the branch as one. **For example**, between summing points, between summing point and takeoff point, between input and summing point, between take-off point and output.

Example

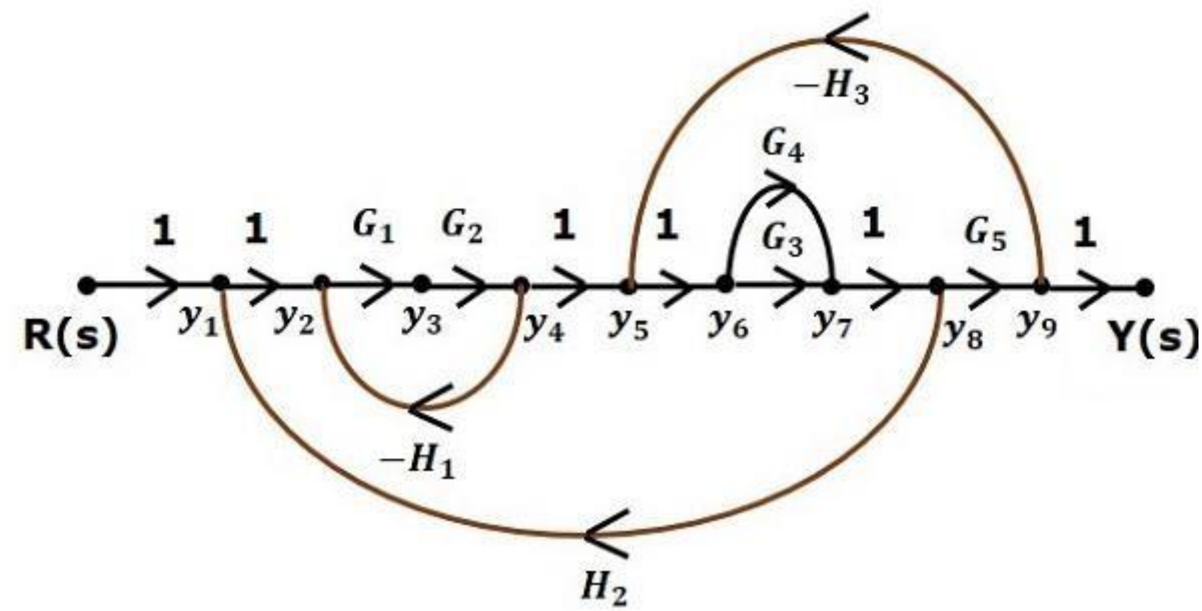
Let us convert the following block diagram into its equivalent signal flow graph.



Represent the input signal R(s) and output signal C(s) of block diagram as input node R(s) and output node C(s) of signal flow graph.

Just for reference, the remaining nodes (y_1 to y_9) are labelled in the block diagram. There are nine nodes other than input and output nodes. That is four nodes for four summing points, four nodes for four take-off points and one node for the variable between blocks $G_1$and G_2.

The following figure shows the equivalent signal flow graph.



Let us now discuss the Mason's Gain Formula. Suppose there are 'N' forward paths in a signal flow graph. The gain between the input and the output nodes of a signal flow graph is nothing but the **transfer function** of the system. It can be calculated by using Mason's gain formula.

Mason's gain formula is

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{N} P_i \Delta_i}{\Delta}$$

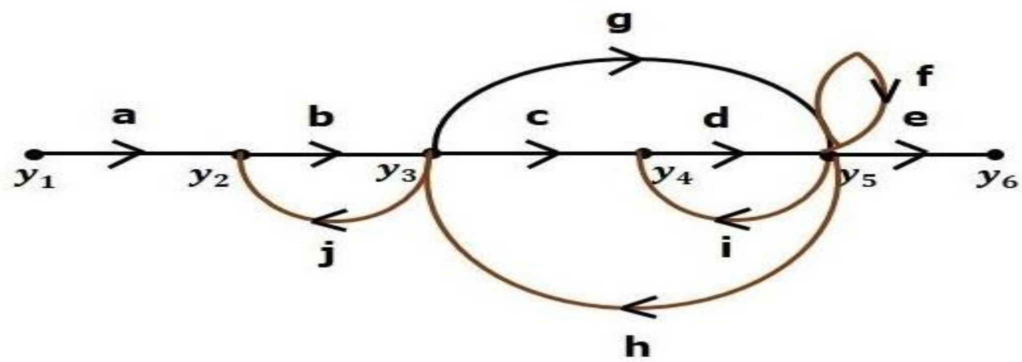
Where,

- **C(s)** is the output node
- **R(s)** is the input node
- **T** is the transfer function or gain between R(s) and C(s)
- **Pi** is the i^th^ forward path gain

  Δ=1−(sum of all individual loop gains) +(sum of gain products of all possible two nontouching loops)−(sum of gain products of all possible three nontouching loops) +....

  Δ_i is obtained from Δ by removing the loops which are touching the i^{th} forward path.

Consider the following signal flow graph in order to understand the basic terminology involved here.



Path

It is a traversal of branches from one node to any other node in the direction of branch arrows. It should not traverse any node more than once.

Examples − $y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5$ and $y_5 \rightarrow y_3 \rightarrow y_2$

Forward Path

The path that exists from the input node to the output node is known as **forward path**.

Examples − $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Forward Path Gain

It is obtained by calculating the product of all branch gains of the forward path.

Examples − $abcde$ is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$ and abge is the forward path gain of $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.

Loop

The path that starts from one node and ends at the same node is known as a **loop**. Hence, it is a closed path.

Examples – $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_3 \rightarrow y_5 \rightarrow y_3$.

Loop Gain

It is obtained by calculating the product of all branch gains of a loop.

Examples – b_j is the loop gain of $y_2 \rightarrow y_3 \rightarrow y_2$ and g_h is the loop gain of $y_3 \rightarrow y_5 \rightarrow y_3$.

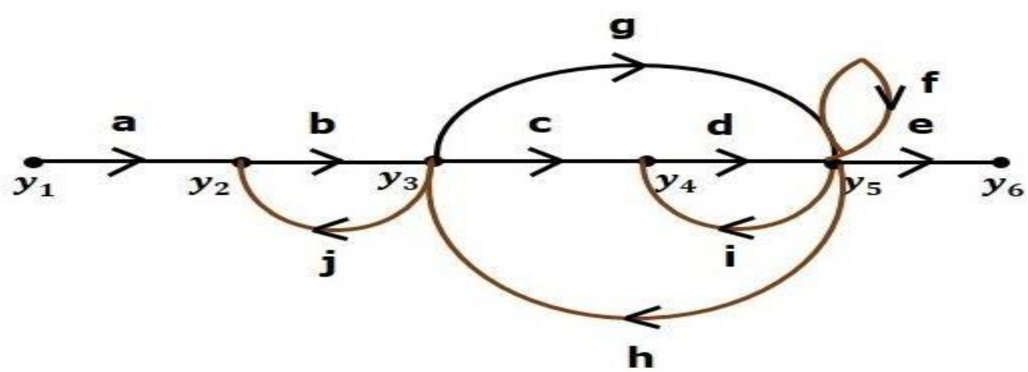
Non-touching Loops

These are the loops, which should not have any common node.

Examples – The loops, $y_2 \rightarrow y_3 \rightarrow y_2$ and $y_4 \rightarrow y_5 \rightarrow y_4$ are non-touching.

Calculation of Transfer Function using Mason's Gain Formula

Let us consider the same signal flow graph for finding transfer function.



- Number of forward paths, N = 2.
- First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.
- First forward path gain, $p_1 = abcde$
- Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$
- Second forward path gain, $p_2 = abge$
- Number of individual loops, L = 5.

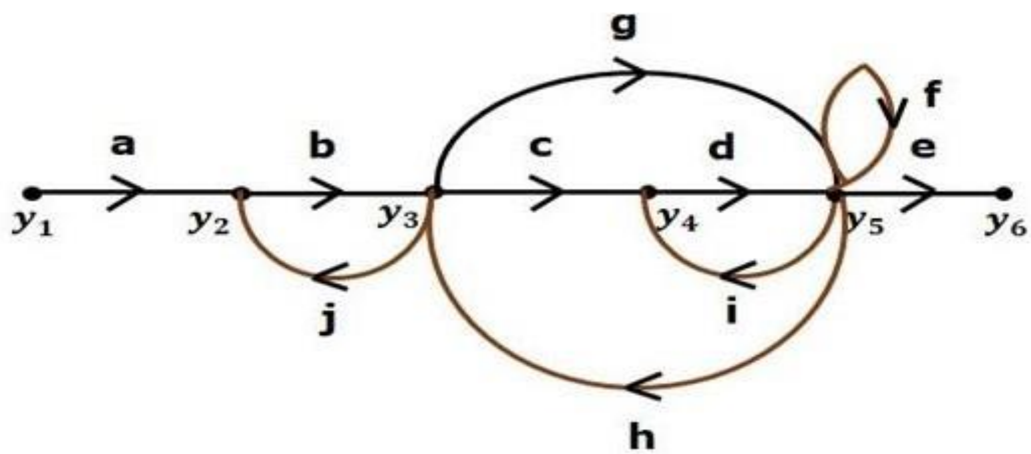
  Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.

  Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.

- Number of two non-touching loops = 2.

- First non-touching loops pair is - $y_2 \to y_3 \to y_2$, $y_4 \to y_5 \to y_4$.
- Gain product of first non-touching loops pair $l_1 l_4 = bjdi$
- Second non-touching loops pair is - $y_2 \to y_3 \to y_2$, $y_5 \to y_5$.
- Gain product of second non-touching loops pair is $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.We know,



- Number of forward paths, N = 2.
- First forward path is - $y_1 \to y_2 \to y_3 \to y_4 \to y_5 \to y_6$.
- First forward path gain, $p_1 = abcde$.
- Second forward path is - $y_1 \to y_2 \to y_3 \to y_5 \to y_6$.
- Second forward path gain, $p_2 = abge$.
- Number of individual loops, L = 5.
- Loops are - $y_2 \to y_3 \to y_2$, $y_3 \to y_5 \to y_3$, $y_3 \to y_4 \to y_5 \to y_3$, $y_4 \to y_5 \to y_4$ and $y_5 \to y_5$.

- Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.
- Number of two non-touching loops = 2.
- First non-touching loops pair is - $y_2 \to y_3 \to y_2$, $y_4 \to y_5 \to y_4$.
- Gain product of first non-touching loops pair, $l_1 l_4 = bjdi$
- Second non-touching loops pair is - $y_2 \to y_3 \to y_2$, $y_5 \to y_5$.
- Gain product of second non-touching loops pair is - $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.

We know,

$$\Delta = 1 - (sum\ of\ all\ individual\ loop\ gains)$$
$$+(sum\ of\ gain\ products\ of\ all\ possible\ two\ nontouching\ loops)$$
$$-(sum\ of\ gain\ products\ of\ all\ possible\ three\ nontouching\ loops) + \ldots$$

Substitute the values in the above equation,

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0)$$

$$\Rightarrow \Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path.

So, $\Delta_1 = 1$.

Similarly, $\Delta_2 = 1$. Since, no loop which is non-touching to the second forward path.

Substitute, N = 2 in Mason's gain formula

$$T = \frac{C(s)}{R(s)} = \frac{\Sigma_{i=1}^{2} P_i \Delta_i}{\Delta}$$

$$T = \frac{C(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

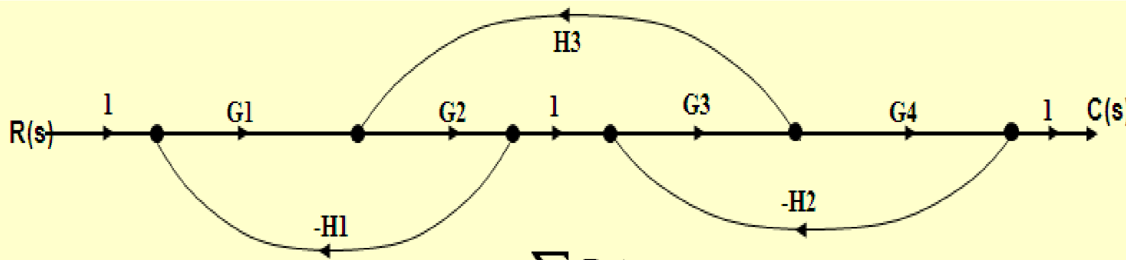
$$T = \frac{C(s)}{R(s)} = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

$$\Rightarrow T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Therefore, the transfer function is -

$$T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

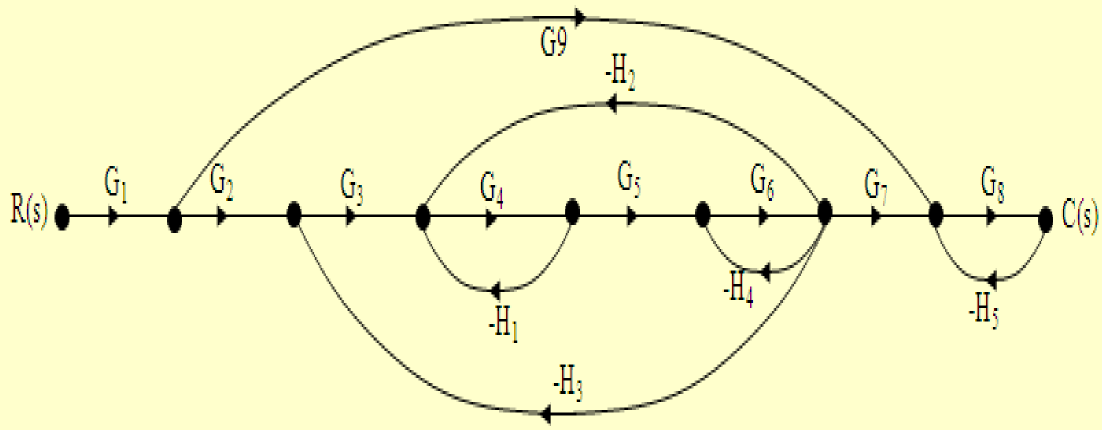
Example-1:



$$T(s) = \frac{\sum P_k \Delta_k}{\Delta}$$

- $P_1 = G_1G_2G_3G_4$ $\quad \Delta_1 = 1$ There is no P_2 or Δ_2 or more.
- $\sum L_1 = -G_1G_2H_1 + G_2G_3H_3 - G_3G_4H_2$
- $\sum L_2 = G_1G_2G_3G_4H_1H_2$
- $\Delta = 1 - \sum L_1 + \sum L_2 = 1 + G_1G_2H_1 - G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2$
- $\mathrm{T(s)} = \frac{\sum \mathrm{P}_1\Delta_1}{\Delta} = \frac{G_1G_2G_3G_4}{1 + G_1G_2H_1 - G_2G_3H_3 + G_3G_4H_2 + G_1G_2G_3G_4H_1H_2}$

Example-2:



$$M_1 = G_1G_2G_3G_4G_5G_6G_7G_8 \qquad \Delta_1 = 1$$

$$M_2 = G_1G_9G_8 \qquad \Delta_2 = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2] + G_4H_1G_6H_4$$

$$= 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4$$

$$\Delta = 1-[-G_4H_1 - G_6H_4 - G_3G_4G_5G_6H_3 - G_4G_5G_6H_2 - G_8H_5]$$

$$+\left[G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3\right]$$

$$+G_4H_1G_6H_4G_8H_5$$

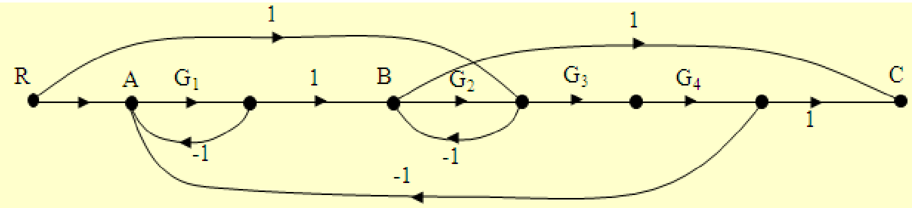
$$\Delta = 1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_8H_5$$

$$+G_4H_1G_6H_4 + G_4H_1G_8H_5 + G_6H_4G_8H_5 + G_8H_5G_4G_5G_6H_2 + G_8H_5G_3G_4G_5G_6H_3$$

$$+G_4H_1G_6H_4G_8H_5$$

$$T(s) = \frac{C(s)}{R(s)} = \frac{M_1\Delta_1 + M_2\Delta_2}{\Delta} = \frac{G_1G_2G_3G_4G_5G_6G_7G_8 + G_1G_9G_8[1 + G_4H_1 + G_6H_4 + G_3G_4G_5G_6H_3 + G_4G_5G_6H_2 + G_4H_1G_6H_4]}{\Delta}$$

Example-3:



$$\mathbf{M_1} = G_1G_2G_3G_4 \qquad \Delta_1 = 1$$
$$\mathbf{M_2} = G_1 \qquad \Delta_2 = 1$$
$$\mathbf{M_3} = G_3G_4 \qquad \Delta_3 = 1 + G_1$$
$$\mathbf{M_4} = -1 \qquad \Delta_4 = 1 + G_1$$
$$\mathbf{M_5} = -G_3G_4G_1 \qquad \Delta_5 = 1$$

$$\Delta = 1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2 = 1 + G_1 + G_2 + G_1G_2G_3G_4 + G_1G_2$$

$$\text{a)} \quad \frac{C}{M} = \frac{G_1G_2G_3G_4 + G_1 + G_3G_4(1+G_1) - (1)(1+G_1) - G_1G_3G_4}{1 - (-G_1 - G_2 - G_1G_2G_3G_4) + G_1G_2}$$

STATE SPACE ANALYSIS OF CONTINUOUS SYSTEMS

The **state space model** of Linear Time-Invariant (LTI) system can be represented as,

$$X^{\cdot}=AX+B$$

$$U$$

$$Y=CX+D$$

$$U$$

The first and the second equations are known as state equation and output equation respectively.

Where,

- X and $X^{\cdot}$ are the state vector and the differential state vector respectively.
- U and Y are input vector and output vector respectively.
- A is the system matrix.
- B and C are the input and the output matrices.
- D is the feed-forward matrix.

Basic Concepts of State Space Model

The following basic terminology involved in this chapter.

State

It is a group of variables, which summarizes the history of the system in order to predict the future values (outputs).

State Variable

The number of the state variables required is equal to the number of the storage elements present in the system.

Examples – current flowing through inductor, voltage across

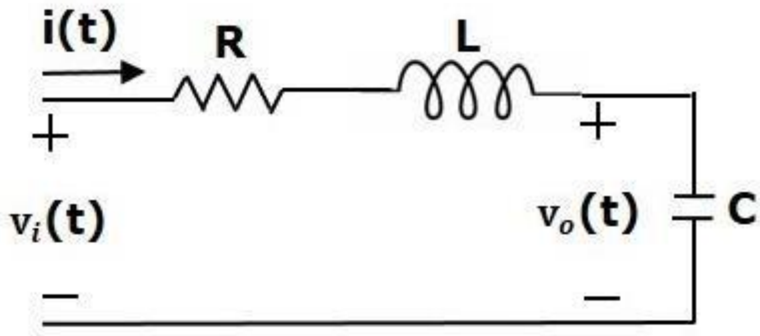
capacitor State Vector

It is a vector, which contains the state variables as elements.

In the earlier chapters, we have discussed two mathematical models of the control systems. Those are the differential equation model and the transfer function model. The state space model can be obtained from any one of these two mathematical models. Let us now discuss these two methods one by one.

State Space Model from Differential Equation

Consider the following series of the RLC circuit. It is having an input voltage, vi(t) and the current flowing through the circuit is i(t).



There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two and these state variables are the current flowing through the inductor, i(t) and the voltage across capacitor, $v_c(t)$.

From the circuit, the output voltage, $v_0(t)$ is equal to the voltage across capacitor, $v_c(t)$.

$$v_0(t) = v_c(t)$$

Apply KVL around the loop.

$$v_i(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + v_c(t)$$

$$\Rightarrow \frac{\mathrm{d}i(t)}{\mathrm{d}t} = -\frac{Ri(t)}{L} - \frac{v_c(t)}{L} + \frac{v_i(t)}{L}$$

The voltage across the capacitor is -

$$v_c(t) = \frac{1}{C}\int i(t)dt$$

Differentiate the above equation with respect to time.

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \frac{i(t)}{C}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

Differential state vector, $\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix}$

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [\, v_i(t) \,]$$

$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix},\ B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix},\ C = [0 \quad 1]\ and\ D = [0]$$

State Space Model from Transfer Function

Consider the two types of transfer functions based on the type of terms present in the numerator.

- Transfer function having constant term in Numerator.
- Transfer function having polynomial function of ‘s’ in

Numerator. Transfer function having constant term in Numerator

Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)Y(s) = b_0 U(s)$$

Apply inverse Laplace transform on both sides.

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1 \frac{\mathrm{d}y(t)}{\mathrm{d}t} + a_0 y(t) = b_0 u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} = \dot{x}_n$$

and u(t)=u

Then,

$$\dot{x}_n + a_{n-1}x_n + \ldots + a_1x_2 + a_0x_1 = b_0u$$

From the above equation, we can write the following state equation.

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + b_0u$$

The output equation is -

$$y(t) = y = x_1$$

The state space model is -

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Here, D=[0].

Example

Find the state space model for the system having transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

Rearrange, the above equation as,

$$(s^2 + s + 1)Y(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + \frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = u(t)$$

Let

$$y(t) = x_1$$

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

and $u(t) = u$

Then, the state equation is

$$\dot{x}_2 = -x_1 - x_2 + u$$

The output equation is

$$y(t) = y = x_1$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Transfer function having polynomial function of ‘s’ in Numerator Consider the following transfer function of a system

$$\frac{Y(s)}{U(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \left(\frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}\right)(b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0)$$

The above equation is in the form of product of transfer functions of two blocks, which are cascaded.

$$\frac{Y(s)}{U(s)} = \left(\frac{V(s)}{U(s)}\right)\left(\frac{Y(s)}{V(s)}\right)$$

Here,

$$\frac{V(s)}{U(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

Rearrange, the above equation as

$$(s^n + a_{n-1}s^{n-1} + \ldots + a_0)V(s) = U(s)$$

Apply inverse Laplace transform on both the sides.

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{n-1}v(t)}{\mathrm{d}t^{n-1}} + \ldots + a_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} + a_0 v(t) = u(t)$$

Let

$$v(t) = x_1$$

$$\frac{\mathrm{d}v((t)}{\mathrm{d}t} = x_2 = \dot{x}_1$$

$$\frac{\mathrm{d}^2 v(t)}{\mathrm{d}t^2} = x_3 = \dot{x}_2$$

.

.

.

$$\frac{\mathrm{d}^{n-1}v(t)}{\mathrm{d}t^{n-1}} = x_n = \dot{x}_{n-1}$$

$$\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} = \dot{x}_n$$

and u(t)=u

Then, the state equation is

$$\dot{x}_n = -a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + u$$

Consider,

$$\frac{Y(s)}{V(s)} = b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0$$

Rearrange, the above equation as

$$Y(s) = (b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0)V(s)$$

Apply inverse Laplace transform on both the sides.

$$y(t) = b_n\frac{\mathrm{d}^n v(t)}{\mathrm{d}t^n} + b_{n-1}\frac{\mathrm{d}^{n-1}v(t)}{\mathrm{d}t^{n-1}} + \ldots + b_1\frac{\mathrm{d}v(t)}{\mathrm{d}t} + b_0 v(t)$$

By substituting the state variables and $y(t) = y$ in the above equation, will get the output equation as,

$$y = b_n\dot{x}_n + b_{n-1}x_n + \ldots + b_1x_2 + b_0x_1$$

Substitute, $\dot{x}_n$ value in the above equation.

$$y = b_n(-a_0x_1 - a_1x_2 - \ldots - a_{n-1}x_n + u) + b_{n-1}x_n + \ldots + b_1x_2 + b_0x_1$$

$$y = (b_0 - b_na_0)x_1 + (b_1 - b_na_1)x_2 + \ldots + (b_{n-1} - b_na_{n-1})x_n + b_nu$$

The state space model is

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} b_0 - b_na_0 & b_1 - b_na_1 & \ldots & b_{n-2} - b_na_{n-2} & b_{n-1} - b_na_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

If $b_n = 0$, then,

$$Y = \begin{bmatrix} b_0 & b_1 & \ldots & b_{n-2} & b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

Transfer Function from State Space Model

We know the state space model of a Linear Time-Invariant (LTI) system is -

$$\dot{X} = AX + B$$

$$U$$

$$Y = CX + D$$

$$U$$

Apply Laplace Transform on both sides of the state equation.

$$sX(s) = AX(s) + BU(s)$$

$$\Rightarrow (sI - A)X(s) = BU(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1}BU(s)$$

Apply Laplace Transform on both sides of the output equation.

$$Y(s) = CX(s) + DU(s)$$

Substitute, X(s) value in the above equation.

$$\Rightarrow Y(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$\Rightarrow Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

$$\Rightarrow Y(s)\ U(s) = C(sI - A)^{-1}B + D$$

The above equation represents the transfer function of the system. So, we can calculate the transfer function of the system by using this formula for the system represented in the state space model.

Note − When D=[0], the transfer function will be

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Example

Let us calculate the transfer function of the system represented in the state space model as,

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad and \quad D = [0]$$

The formula for the transfer function when $D = [0]$ is -

$$\frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

Substitute, A, B & C matrices in the above equation.

$$\frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}}{(s+1)s - 1(-1)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s \\ 1 \end{bmatrix}}{s^2 + s + 1} = \frac{1}{s^2 + s + 1}$$

Therefore, the transfer function of the system for the given state space model is

$$\frac{Y(s)}{U(s)} = \frac{1}{s^2 + s + 1}$$

State Transition Matrix and its Properties

If the system is having initial conditions, then it will produce an output. Since, this output is present even in the absence of input, it is called **zero input response** $x_{ZIR}(t)$. Mathematically, we can write it as,

$$x_{ZIR}(t) = e^{At}X(0) = L^{-1}\left\{[sI - A]^{-1}X(0)\right\}$$

From the above relation, we can write the state transition matrix ϕ(t) as

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

So, the zero input response can be obtained by multiplying the state transition matrix ϕ(t) with the initial conditions matrix.

Properties of the state transition matrix

- If t=0, then state transition matrix will be equal to an Identity matrix.

ϕ(0)=I

- Inverse of state transition matrix will be same as that of state transition matrix just by replacing 't' by '-t'.

$$\phi^{-1}(t) = \phi(-t)$$

- If t=t1+t2 , then the corresponding state transition matrix is equal to the multiplication of the two state transition matrices at t=t1t=t1 and t=t2t=t2.

ϕ(t1+t2)=ϕ(t1)ϕ(t2)

Controllability and Observability

Let us now discuss controllability and observability of control system one by

one. Controllability

A control system is said to be **controllable** if the initial states of the control system are transferred (changed) to some other desired states by a controlled input in finite duration of time.

We can check the controllability of a control system by using **Kalman's test**.

- Write the matrix Qc in the following form.

$$Q_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

- Find the determinant of matrix QcQc and if it is not equal to zero, then the control system is controllable.

Observability

A control system is said to be **observable** if it is able to determine the initial states of the control system by observing the outputs in finite duration of time.

We can check the observability of a control system by using **Kalman's test**.

- Write the matrix Qo in following form.

$$Q_o = \begin{bmatrix} C^T & A^TC^T & (A^T)^2C^T & \dots & (A^T)^{n-1}C^T \end{bmatrix}$$

- Find the determinant of matrix QoQo and if it is not equal to zero, then the control system is observable.

Example

Let us verify the controllability and observability of a control system which is represented in the state space model as,

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [u]$$

$$Y = [0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here,

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad [0 \quad 1], D = [0] \quad and \quad n = 2$$

For $n = 2$, the matrix Q_c will be

$$Q_c = [B \quad AB]$$

We will get the product of matrices A and B as,

$$AB = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow Q_c = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$|Q_c| = 1 \neq 0$$

Since the determinant of matrix Qc is not equal to zero, the given control system is controllable.

For n=2, the matrix Qo will be –

$$Q_o = [C^T \quad A^T C^T]$$

Here,

$$A^T = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \quad and \quad C^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We will get the product of matrices A^T and C^T as

$$A^T C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow Q_o = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\Rightarrow |Q_o| = -1 \quad \neq 0$$

Since, the determinant of matrix Qo is not equal to zero, the given control system is observable.Therefore, the given control system is both controllable and observable.