CE3203PC: STRUCTURAL ANALYSIS-II

Topic Name: Two Hinged Arches

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ANALYSIS OF TWO-HINGED ARCHES

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A typical two-hinged arch is shown in Figure. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two-hinged arch.

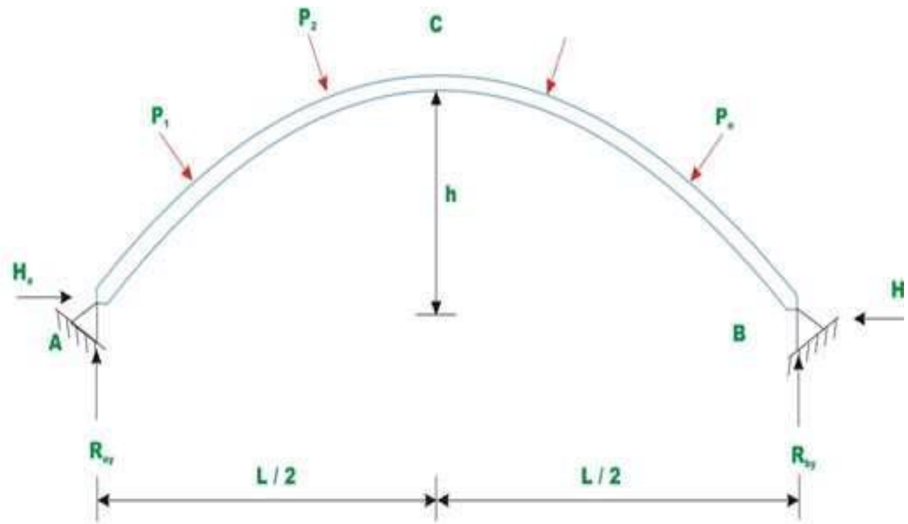


Fig. a Two-Hinged Arch

Symmetrical Two Hinge

Figure shows a two-hinged arch hinged only at the abutments A and B. The vertical reactions R_a and R_b ,of the course, may be determined by taking moments about either hinge.

The horizontal thrust at each support may be determined from the condition that the horizontal displacement of either hinge with respect to the other is zero.

Let M be the beam moment at any section X.

Actual bending moment at the section is given by

$$M_x = (M-H_y)$$

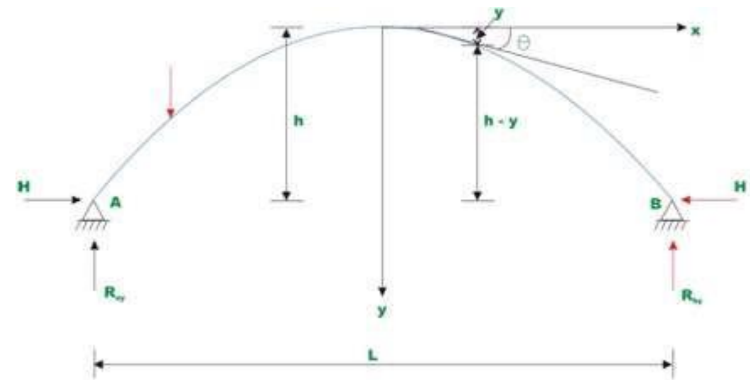


Fig.a

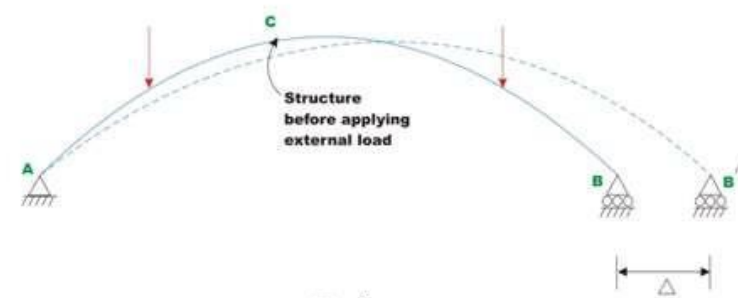


Fig. b

Therefore, total strain energy stored by the whole arch is

$$W_i = \int M_x^2 \, ds / 2EI$$

$$= \int (M - Hy)^2 \, ds / 2EI$$

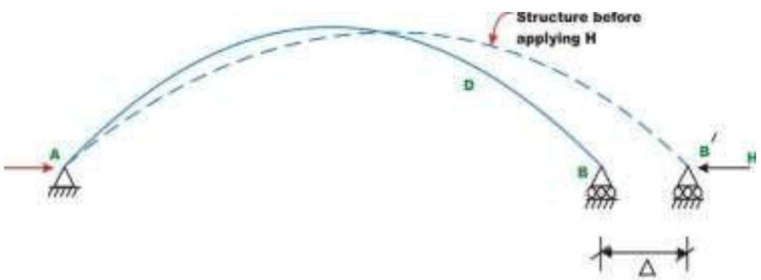


Fig. c

By the first theorem of castigliano the horizontal end relative to other is given by $\partial W_i / \partial H$.

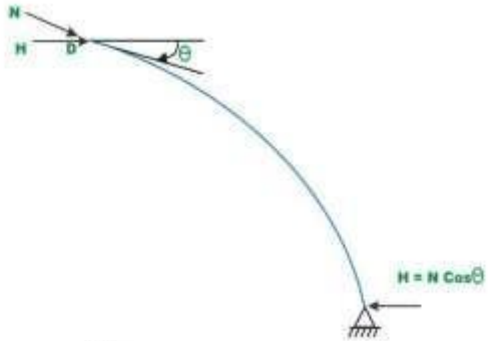


Fig. d

Since such a relative horizontal displacement of one end with respect to the other end is not possible in two hinged arch.

$$\partial W_i / \partial H = 0$$

$$\partial W_i/\partial H = \int 2(M - Hy)(-y)\,ds/2EI$$

$$= \int Myds/2EI - H\int y^2 ds/2EI = 0$$

$$\mathbf{H = \int Myds/2EI / \int y^2 ds/2EI}$$

If the arch is of uniform flexural rigidity EI,

$$\mathbf{H = \frac{\int Myds}{\int y^2 ds}}$$

TEMPERATURE EFFECT

Consider an unloaded two-hinged arch of span L. When the arch undergoes a uniform temperature change of T ° C , then its span would increase by α LT if it were allowed to expand freely (Fig a). α is the co-efficient of thermal expansion of the arch material. Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increased.

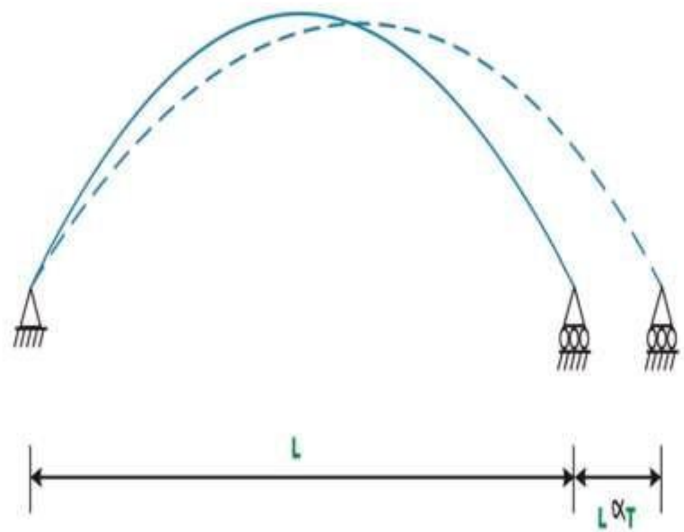


Fig. a

Considering the end B as a roller end with an external horizontal force H applied at B, the bending moment at any section is given by

$$M_x = -Hy$$

Strain energy stored by the arch $= W_i = \int M_x^2 ds/2EI$

$$= \int H^2 y^2 ds/2EI$$

By the first theorem of castigliano,

Inward horizontal movement of B $= \delta$

$$\delta = \partial W_i/\partial H$$

$$= \int 2H(y^2 ds/2EI)$$

$$\delta = H \int y^2 ds/EI$$

The condition that H may represent the horizontal thrust for the two hinged arch subjected to the rise of temperature is,

$$\delta = \alpha T I$$

$$H \int y^2 ds/EI = \alpha T I$$

$$H = \alpha T I / \int y^2 ds/EI$$

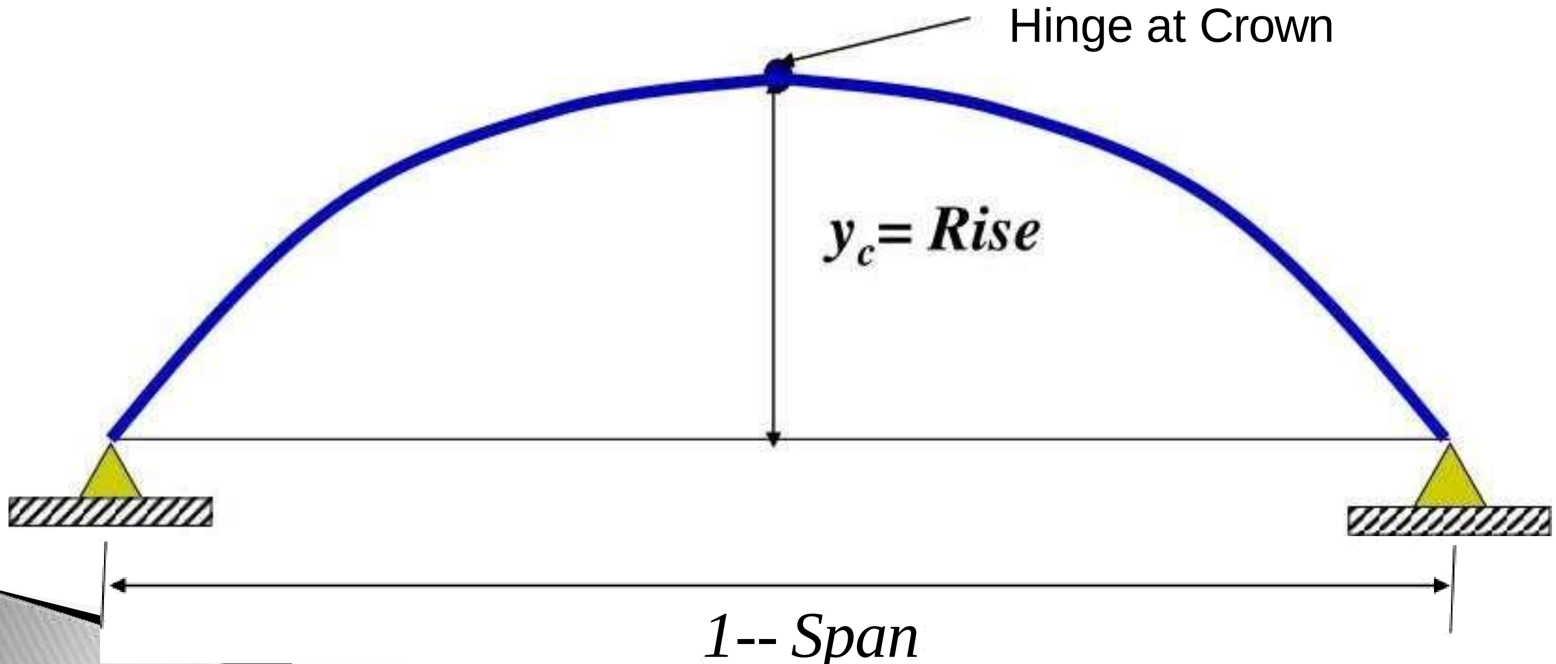
the arch section is of uniform flexural rigidity,

$$\mathbf{H = EI\, \alpha T I / \int y^2 ds}$$

- Arch is a curved structure or humped beam, primarily bears the applied loads by compression.
- The hinge introduced anywhere in the arch makes the structure determinate as
  - Both supports are assumed to be hinged
  - Hinge introduced in the arch provides a further equation to analyze the arch i.e., moment of all forces about hinge is equal to zero.

Introduction

- Normally the third hinge is introduced at the top most point on the arch curve known as crown.



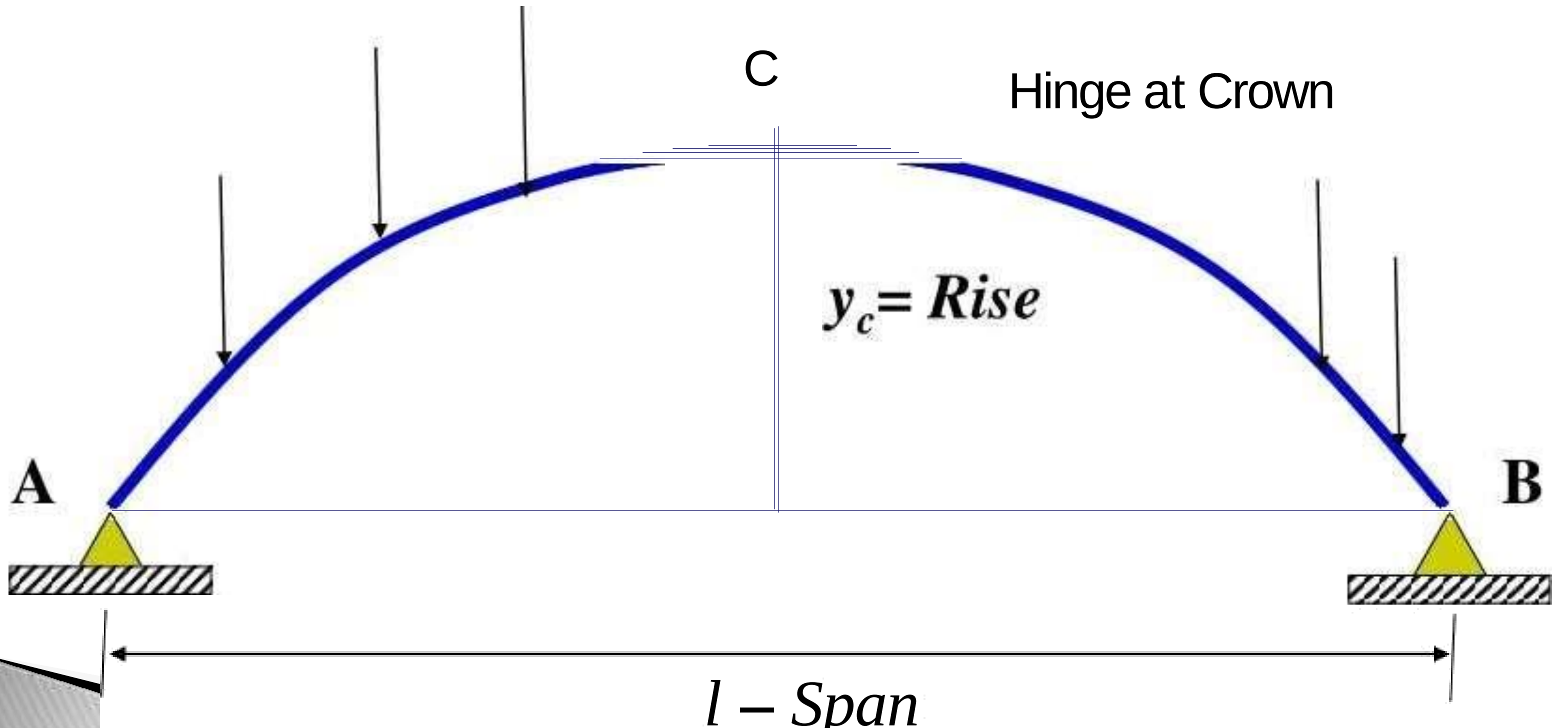
Introduction

- One can solve the arch as beam if we know the horizontal reactions at various supports.

- With the help of third hinge, we can easily determine the horizontal reactions and hence the arch can be analyzed.

Arch Formula's

- Consider an arch ACB, hinged at A, B and C.
- / is the horizontal span and y_c is central rise.



Arch Formula's

- Obtain the vertical reactions Vy and V_B at the ends as usual. To find the horizontal thrust, MC the moment at central hinge must be zero.

General Derivation

- To find *H* the horizontal thrust, M the moment at central hinge C must be known.

$$3f, = y, + H.y, - 0$$

$$H \quad \text{———} \quad \text{——————} (1)$$

$$X.-$$

- We can find moment at any cross-section X of the arch whose coordinates are (x, y).

$$M, \quad - y \quad + Hy \quad \text{——} \quad - (2)$$

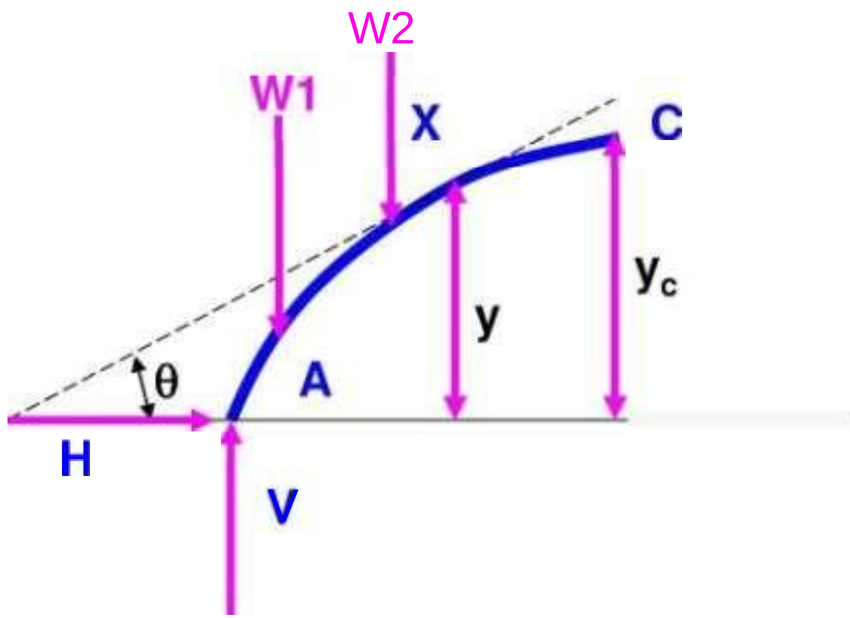
General Derivation

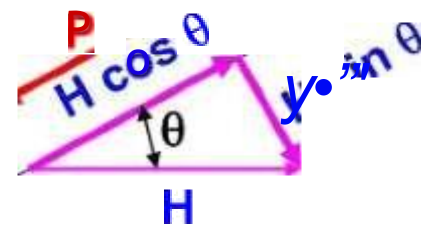
- The vertical and horizontal actions on the section, considering the portion AX arc,
- Vertical reaction can be find out by vertical shear force at the section as for a straight horizontal beam.
- Horizontal thrust at both ends is same.

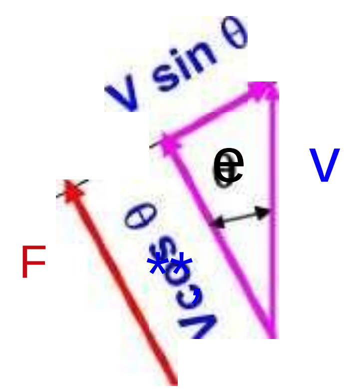
$$= \quad -W1 - \mathrm{JV2}$$

$$H = H$$

General Derivation







$$P = H\cos\theta + V\sin\theta \qquad (3)$$

$$F = H\sin\theta - V\cos\theta \qquad (4)$$

Æ0

General Derivation

- Draw the tangent at X to the centre-line of the arch and let its inclination to the horizontal must be 6.
- Resolving V and H normally to the section and tangentially, i.e., along the tangent at X)
- If the resultant T is required, use eq.5

$$P \quad C \cos d + V \sin k \text{ ——————— } (3)$$

$$F \text{ —} / \quad \sin 6 \text{—} V \cos 6 \text{ ——————— } (4)$$

$$T \quad \sqrt{V^2 + H^2} .. or .. \sqrt{P^2 + F'} \text{ —— } (S)$$

Parabolic Arch

- If the three hinged arch is parabolic in shape and if it carries a uniformly distributed load over the entire span,
  - every action of the arch will be purely in compression,
  - » resisting only a normal thrust;
  - there will be no shear force nor B. M. at the section.

Parabolic Arch

- The linear arch for a given load system on an arch represents the y-diagram.

x With a uniformly distributed load over the entire span, the y-diagram is a parabola.

- The linear arch which is parabolic will, then, have three points (at the hinges) in common with the centre-line of the actual arch, which is also parabolic.

Parabolic Arch

- The linear arch will therefore be identical with the actual arch.

x For any other loading on a parabolic arch, there will be three straining actions, P , F and M at any section.

- To obtain the bending moment, it will be necessary to calculate the rise at any section of the arch.

x —— , y ——

$y,$

It may also be noted that at quarter points, where z=//’4, the *rize* ^⁄4 y,

2 . 2

4y

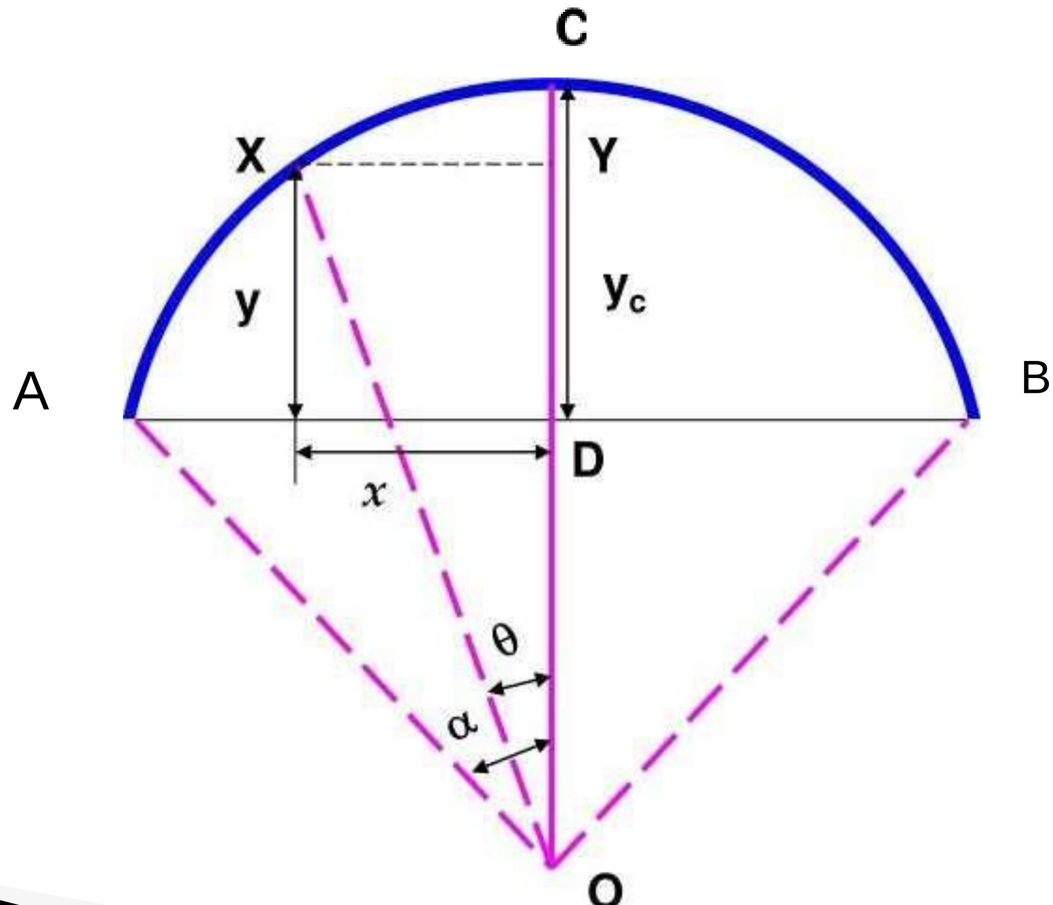
4y . $x\{l — x)$

J 2

$d›$ - tan8= 4/. (/ — 2x)

dx

Circular Arch



Circular Arch

- If the centre-line of the arch is a segment of a circle of radius R, it is more convenient to have the origin at D, the middle of the span. Let (x, y) be the coordinates of a section.

Circular Arch

$AO \quad OB \quad R$

$AD — DB \quad \frac{'}{2}$

$DC — y$

$OY — OD + DY — R —)+ y$

•y =•

$07€^2 = OY^2 + XY$

$$R^2 = x^2 + \{(R - y_c) + y\}^2$$

$y\ (2A — y,) = \frac{l^2}{4}$

$\sin o \quad \frac{AD}{AO} \quad \frac{1}{2A}$

$cOST \quad \underline{OD} \quad \underline{(A — y)}$

$x = OX \sin 8 — /t \sin Ø$

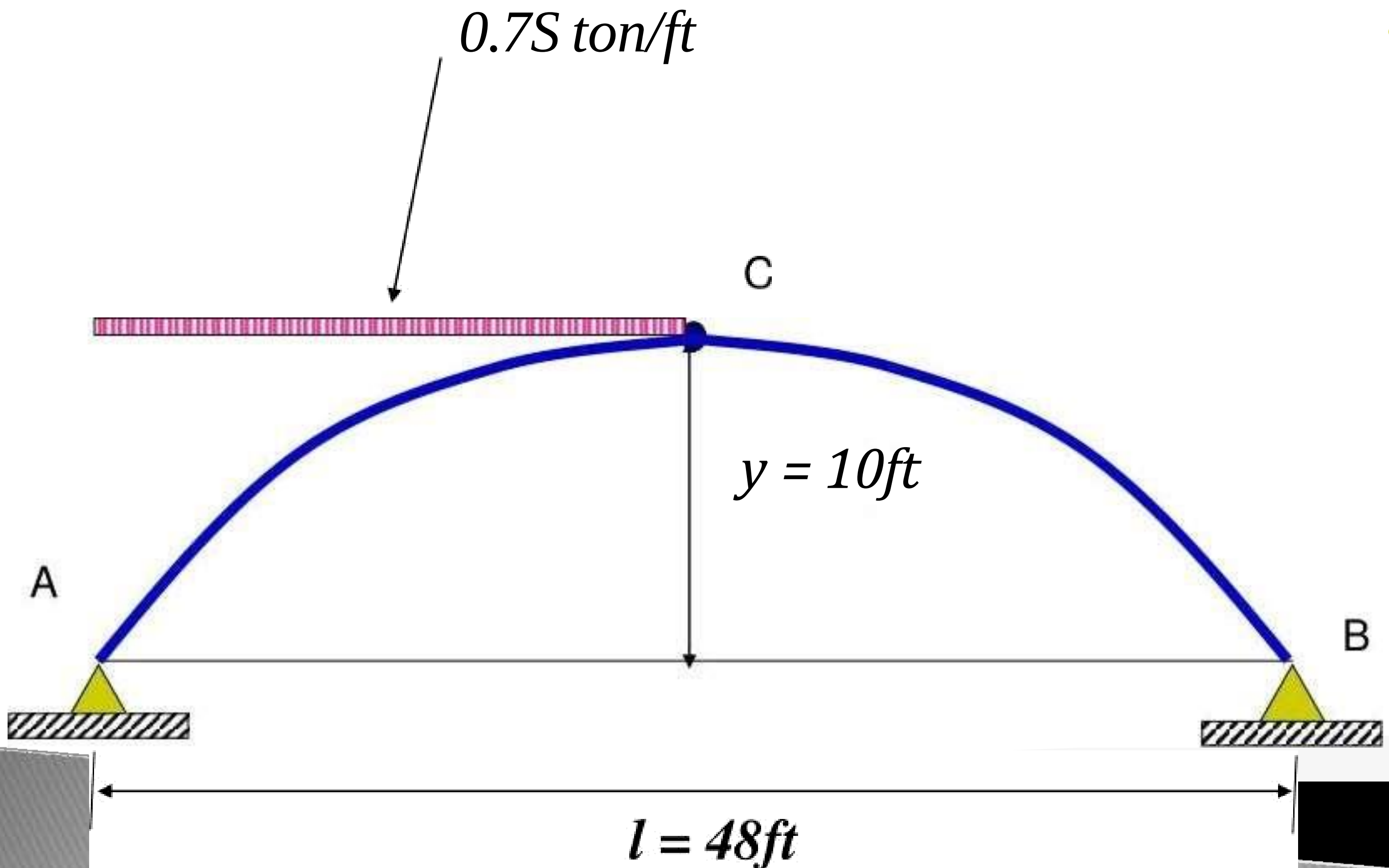
$y = OY — OD — R \cos Ø — A c o s o$

$y' = /t(«o»B — єoso)$

Æ8

Example 01

- A three hinged Parabolic arch, at the crown and springings has a horizontal span of 48ft. It carries UDL of .75 Ton/ft run over the left hand half of the span. Rise = 10'. Calculate the
  - Reactions
  - Normal Thrust
  - Shear force and BM at 6, 12, 30ft from left hinge.


0.7S ton/ft
C
y = 10ft
A
B
l = 48ft

Solution

$V_A = 13.5$ T; $V_B = 4.5$
T; $H = 10.8$ T

y = 5/288*x(48-x), 0 = 5/288*(48-2x)

x	y	0	cos0	sin8	M	V	P	F
6	4.375	32º	0.848	0.529	-20.25	9	13.93	1.91
12	7.5	22○37'	0.923	0.348	-27	4.5	11.7	0
30	9.66	11°46’	0.979	0.203	20.25	4.5	11.49	2.2

Example 02

- Circular arch of span 80 ft with central rise 16 ft is hinged at crown and supports. Carries a point load of 10 tons 20ft from left support.
  - Reactions
  - Normal Thrust
  - Maximum and Minimum BM

Solution

$V_A = 7\cdot5$

T; $V_B =$

2.5 T; **H** =

6.25 T

/4t

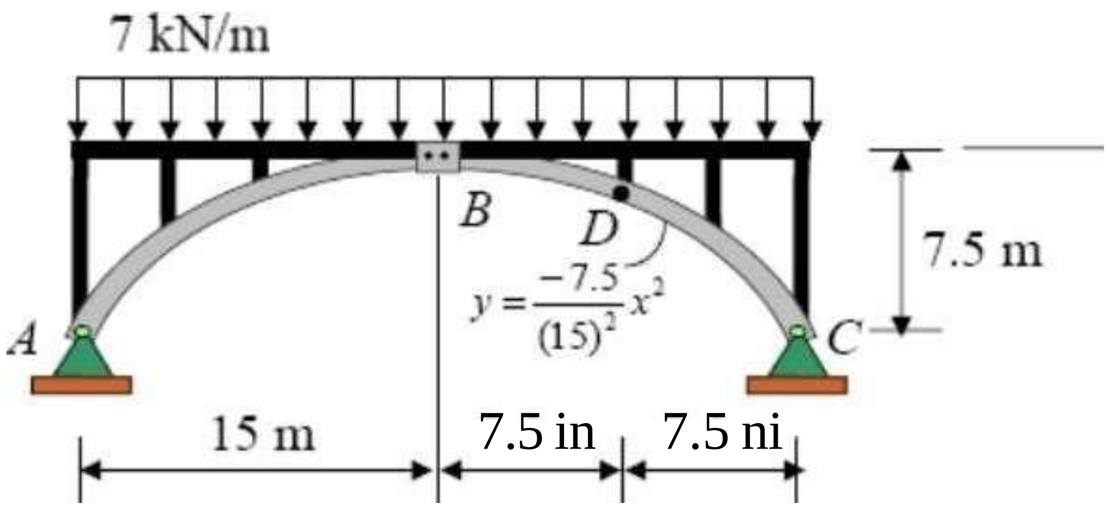
y,(2R−Yc)−

R^2=x ^{2}+{y1(R c)t^2

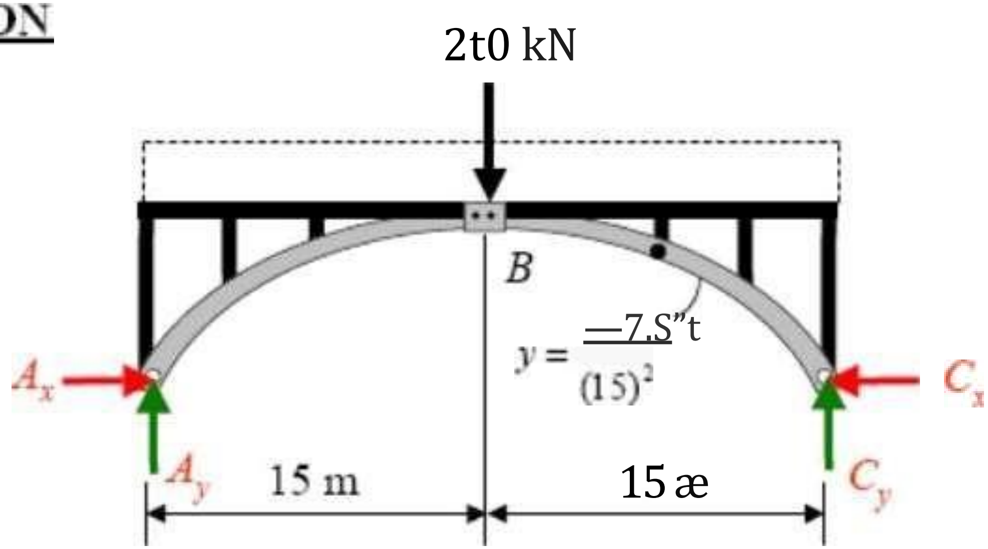
Example 6

The three-hinRcd open-spaudrcl arch bridpc shown iii the figure bclou has a parabolic shape and supports the unifomi load . Show that the parabolic arch is subjected only to axial compression at an intermediate point D along its axis.

Assume the load is iuiifoivily tmnsmitted to the arch ribs.



SOLUTION



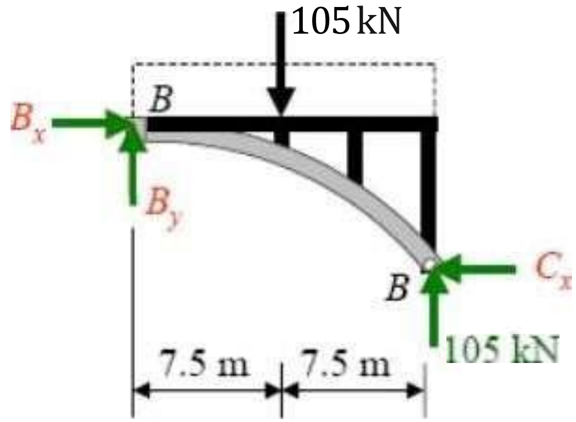
Entire arch :

+↺ $\Sigma M_A = 0$:   C,,(30)—210(15) 0

C ,—105 IN

+Î £P = 0: d   ,—210+105= 0

At,—t05 kN



Arch segment *BC :*

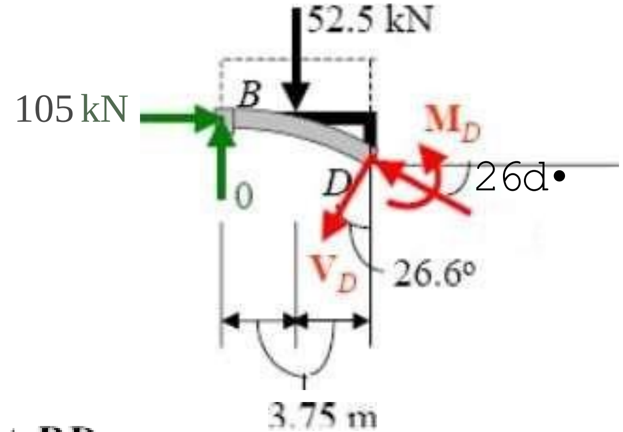
$$\curvearrowright+ \ \Sigma M_B = 0: \qquad -105(7.5)+105(15)-C_x(7.5)=0$$

$$C_x = 105 \text{ kN}$$

ZF,−0: B_E —105 k.Y

+Î ZF$_{\rangle}$ = 0: $B_{\rangle}$ —105+ 105 —0

B —0



Arch segment *BD*

:

Asection of the arch taken through point *D*. x —7.5 m y = -7.5(7.5*)/(t5)* = -1.875 m: is shown in the figure. The slope of the segment at *D* is

$$\tan\theta = \frac{dy}{dx} = \frac{-15}{(15)^2} x\Big|_{x=7.5} = -0.5 ,$$ 0= 26.6s

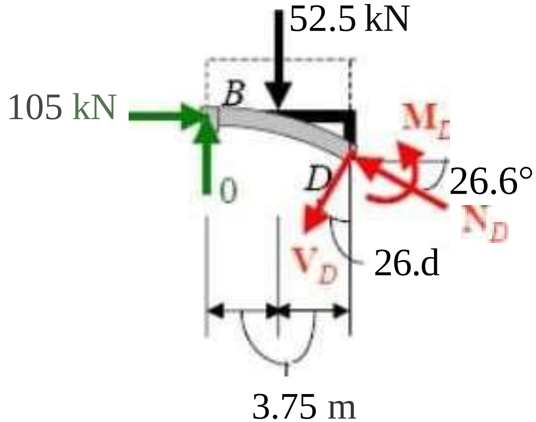
$\xrightarrow{+} \Sigma F_x = 0$: 105 - *Np* cos 26.6° - Kpsin 26.6e = 0

<*f* . —0: -52.5 + *Np* sin 26.6° - Kpcos 26,go 0

$\circlearrowleft + \Sigma M_D = 0$: *MQ* + 52.5(3.75) - 105(1.875) = 0

N —117.40 kN. *V* —0, I —0 kN

Alternate Method



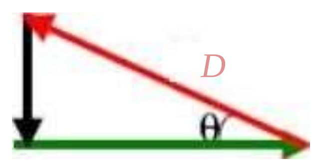
Arch segment *BD* :

A section of the arch taken throiigb point *D*. *x* - 7.5 m, y - -7.5(7.5*)/(15)² = -1.875 m,

iS shoWn in tlle figure. T6e Slope Of the Segment at *D E*

$$\tan\theta = \frac{dy}{dx} = \frac{-15}{(15)^2} x\Big|_{x=7.5} = -0.5 \qquad 6 = 26.6^\circ$$

7.5 iv = (7.5)(7)= 52.5 kN



Nq=25 kN

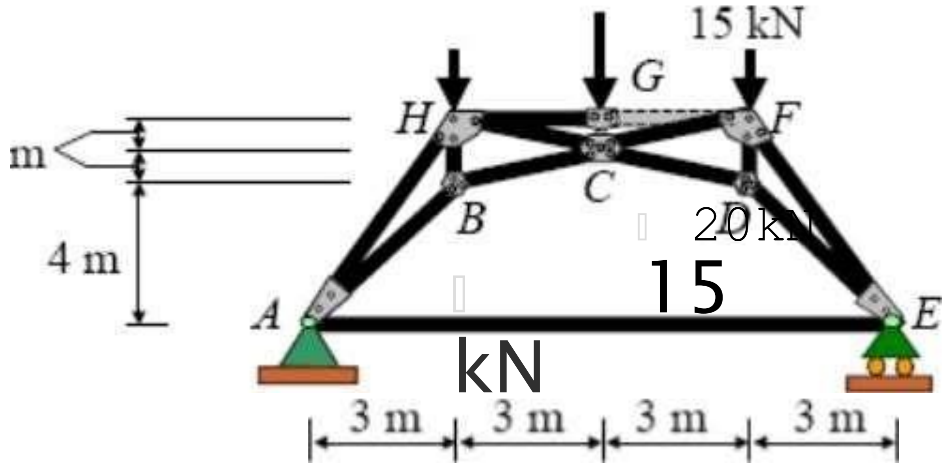
T = *Tg* = (1 5) + (2 5)

T_{max} = 117.4 kN

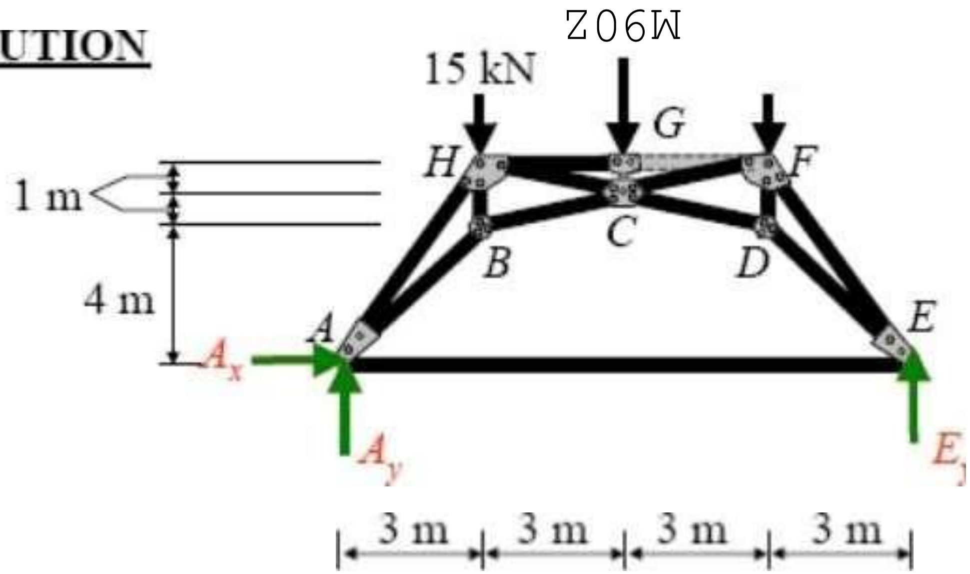
Notes : Since the arch is a parabola. their arc no shear acid bending moment, ooly I_D *E* present

Example S-7

 The three-hinged tied arch is subjected to the loading shown in the figure below. Detemiine the force in members CA rind *CB*. Tb*c* dashed member GF of the trust is intended to carry no force.



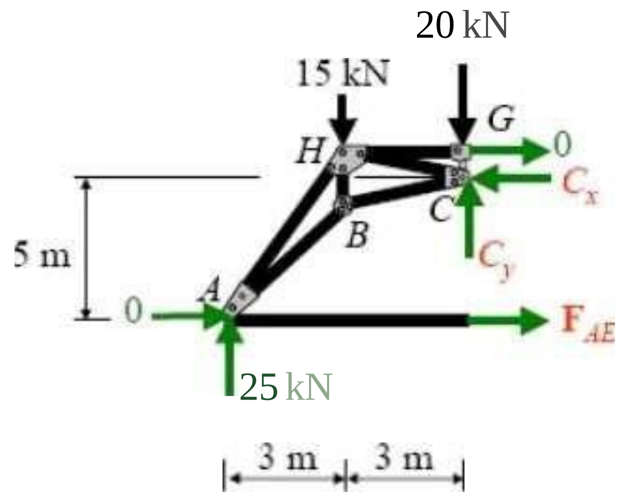
SOLUTION



$$\circlearrowright + \ \Sigma M_A = 0: \qquad E_y(12) - 15(3) - 20(6) - 15(9) = 0$$

$$E_y = 25 \text{ kN}$$

$$A_y = 25 \text{ kN}$$



-+) - 0: N (5) — 25(6)+15(3) = 0

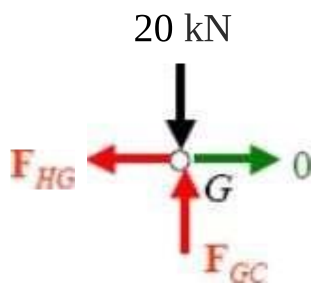
F -- 21.0 kN

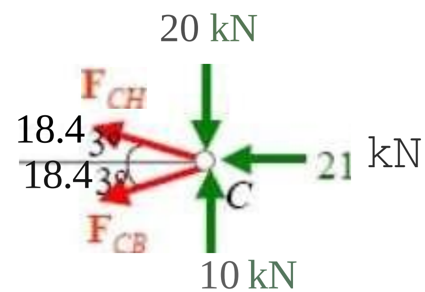
ZF,=0: -Cg + 21= 0

Cq - 21.0 kN

$+\uparrow \Sigma F_y = 0$: 25 — 15 — 20+ C' = 0

Cl — 10 kN





Joint G :

$\xrightarrow{+} \Sigma F_x = 0$: $\quad F_{HG} = 0$

$+\uparrow \Sigma F_y = 0$: $\quad F_{GC} - 20 = 0$

F -- 20 kN (C)

Joint C :

−*ZF,=0:

-N cos18.43 - N cos18.43 - 21= 0

+Î ZF = 0:

N sinl8.43 - Ny cosl8.43 - 20 10 = 0

Thus,

F -- 4.75 kN (T).

F - -26.88 kN (C)

1.2 Moment Distribution method

Outline of the presentation

- Introduction to moment distribution method.
- Important terms.
- Sign conventions.
- Fixed end moments (FEM)
- Examples;
- (A) example of simply supported beam
- (B) example of fixed supported beam with sinking of support.

- The moment distribution method was first introduced by Prof. Hardy Cross of Illinois University in 1930.
- This method provides a convinient means of analysing statically indeterminate beams and rigid frames.
- It is used when number of reduntants are large and when other method becomes very tedious.

Important terms

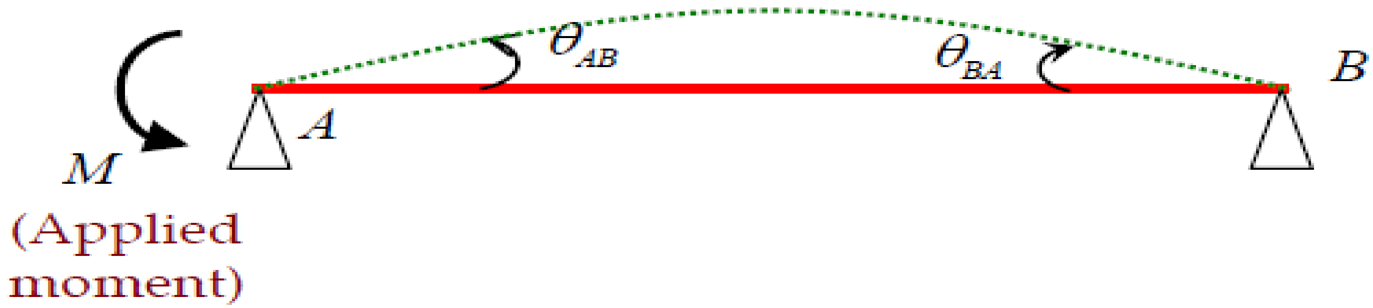
1. Stiffness

The moment required to produce a unit rotation (slope) at a simply supported end of a member is called Stiffness. It is denoted by 'K'.

A) Stiffness when both ends are hinged.

B) Stiffness when both ends are fixed.

A) Beam hinged at both ends:



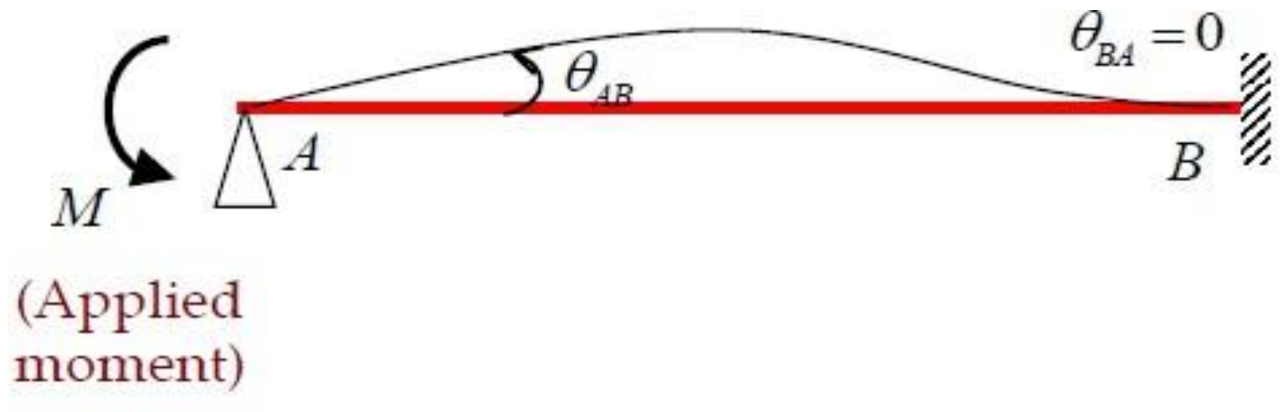
$$\frac{M_{AB}}{\theta_{AB}} = \frac{3EI}{L}$$

i.e., the moment required at A to induce a unit rotation at A is $\frac{3EI}{L}$

(when the far end B is free to rotate)

This moment, i.e., moment required to induce a unit rotation, is called stiffness (denoted by k).

B) Beam hinged at near end and fixed at far end:



$$M_{AB} = \frac{2EI}{L}(2\theta_{AB} + 0) \qquad \Rightarrow \frac{M_{AB}}{\theta_{AB}} = \frac{4EI}{L}$$

i.e., the moment required at A to induce a unit rotation at A is $\frac{4EI}{L}$

(when the far end B is fixed against rotation)

A moment applied at the near end induces at a fixed far end a moment equal to half its magnitude, in the same direction.

Half of moment applied at the near end is carried over to the fixed far end.

Carry over factor is 1/2.

Cont..

Distribution factor (D.F.)

- The factor by which the applied moment is distributed to the member is known as the distribution factor.
- Figure:

- far-end pined (DF = 1)
- far-end fixed (DF = 0)

-

Cont..

Several members meeting at a joint

$$M_1 = \frac{3E_1I_1}{L_1}\theta = k_1\theta$$

$$M_2 = \frac{4E_2I_2}{L_2}\theta = k_2\theta$$

$$M_3 = \frac{3E_3I_3}{L_3}\theta = k_3\theta$$

$$M_4 = \frac{4E_4I_4}{L_4}\theta = k_4\theta$$

$$M_1 : M_2 : M_3 : M_4 :: k_1 : k_2 : k_3 : k_4$$

Cont….

$$M_1 = \frac{k_1}{k_1 + k_2 + k_3 + k_4} M = \frac{k_1}{\sum k} M$$

$$M_i = \frac{k_i}{\sum k} M$$

A moment applied at a joint, where several members meet, will be distributed amongst the members in proportion to their stiffness.

$$M_i = \underbrace{\boxed{\frac{k_i}{\sum k}}}_{\text{distribution factor}} M$$

Sign Conventions

A) Support moments :

clockwise moment = +ve

anticlockwise moment = -ve

B) Rotation (slope):

clockwise moment = +ve

anticlockwise moment = -ve

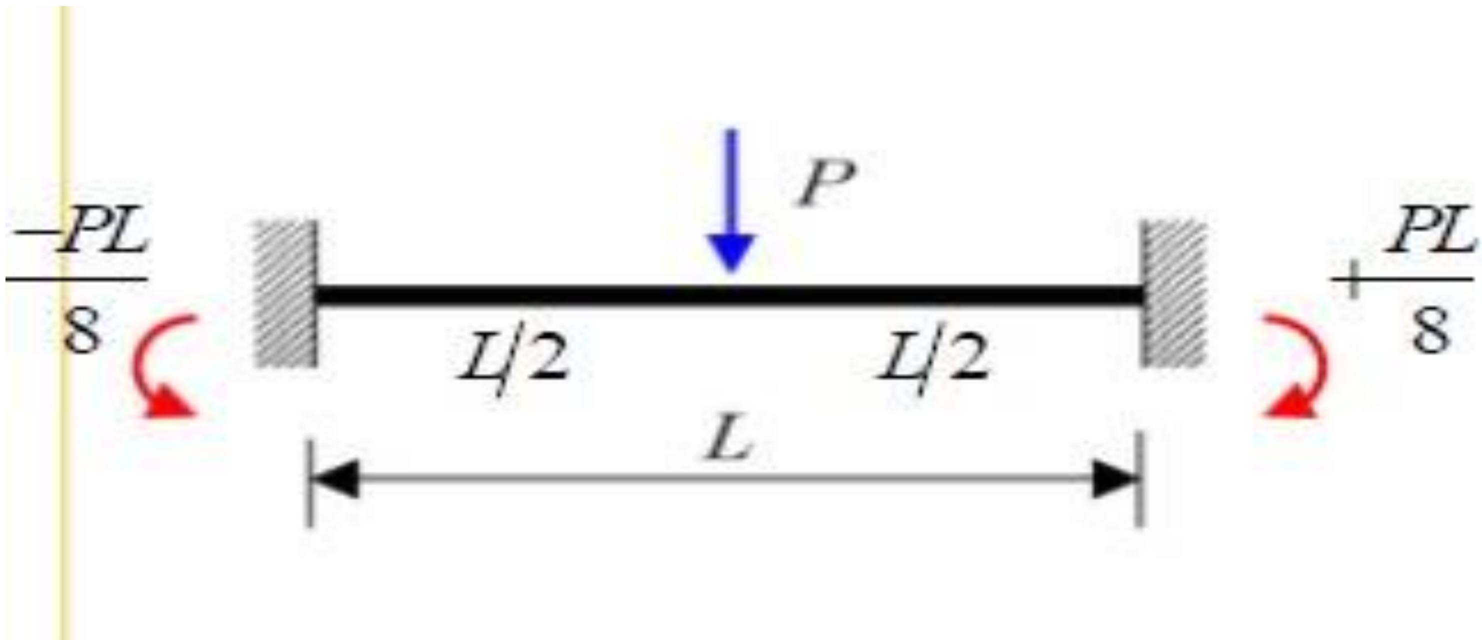
Cont...

C) Sinking (settlement)

- The settlement will be taken as +ve, if it rotates the beam as a whole in clockwise direction.
- The settlement will be taken as -ve, if it rotates the beam as a whole in anti-clockwise direction.

Fixed End Moments

- The fixed end moments for the various load cases is as shown in figure;
- a) for centric loading;



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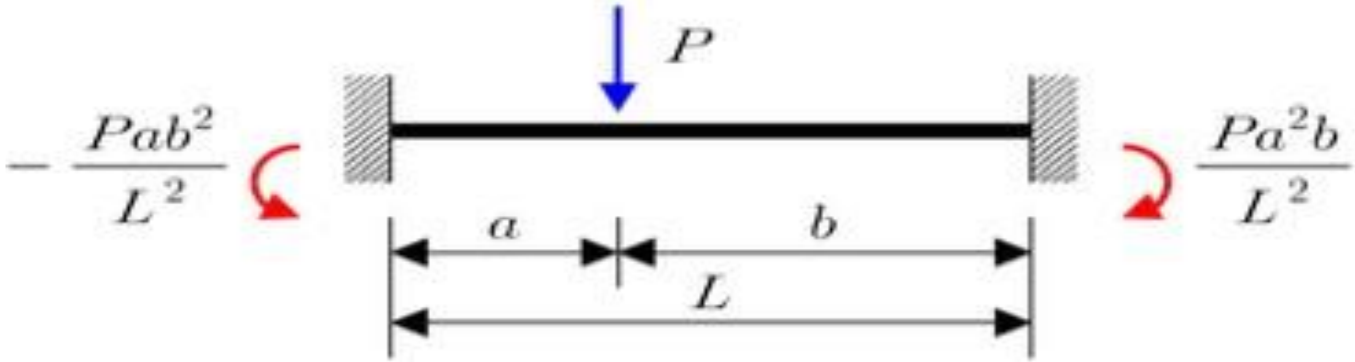
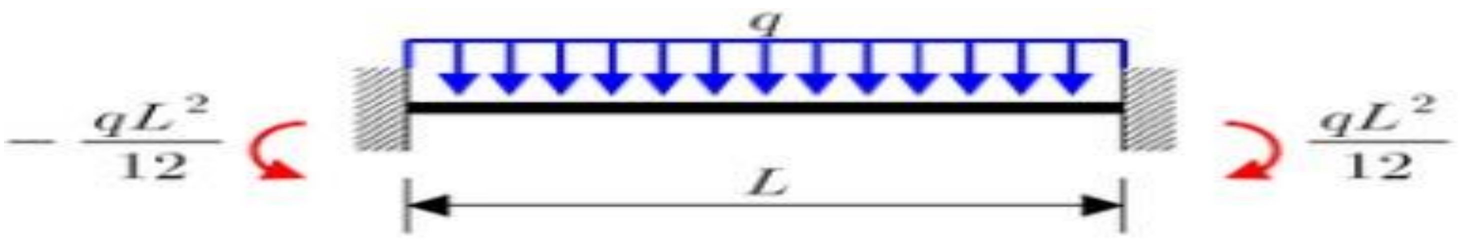
b) for eccentric loading, udl,rotation,sinking of supports & uvl





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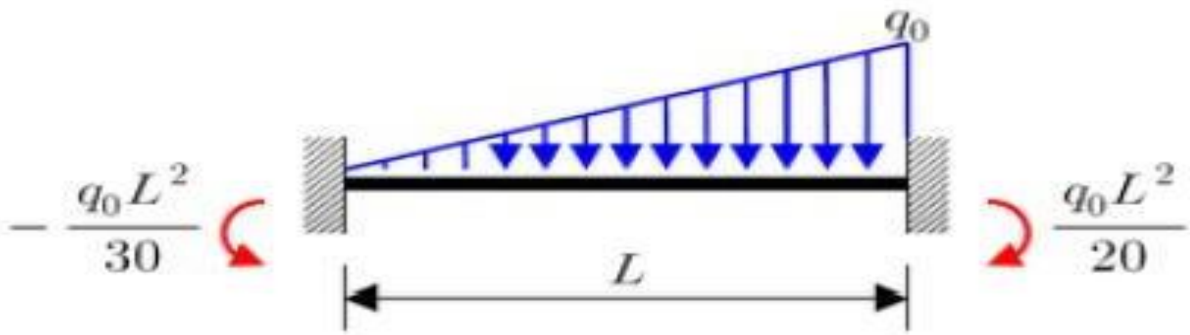
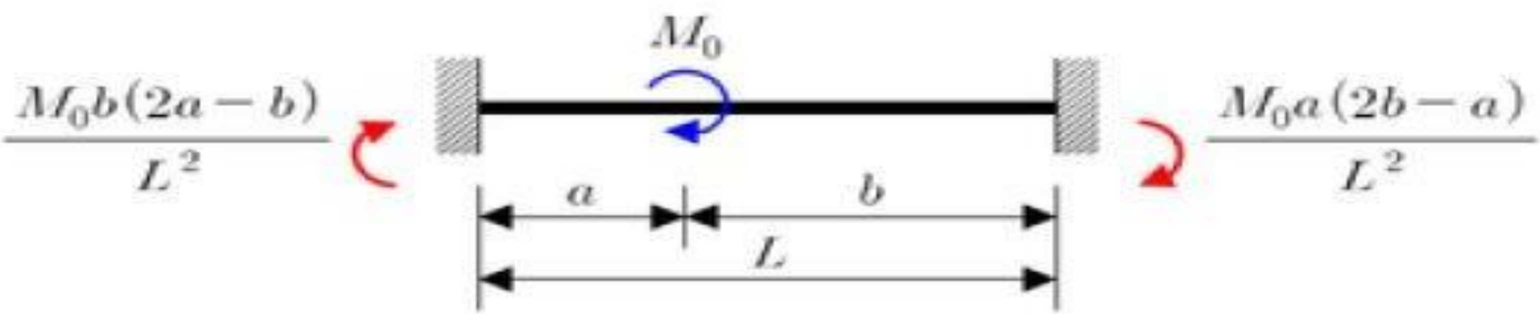
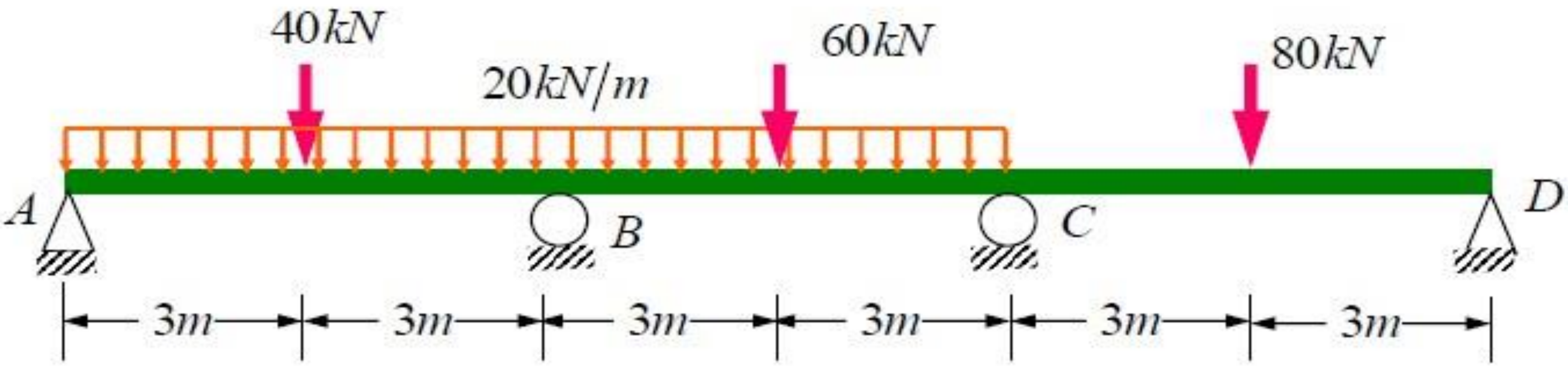

q_0
$-\frac{q_0 L^2}{30}$
$\frac{q_0 L^2}{20}$
L


M_0
$\frac{M_0 b(2a-b)}{L^2}$
$\frac{M_0 a(2b-a)}{L^2}$
a
b
L

Cont...

- Fixed end moment for sinking of supports : $-\dfrac{6EI\delta}{L^2}$

Example 1



Fixed end moments

$$-FEM_{AB} = FEM_{BA} = \frac{wl^2}{12} + \frac{Pl}{8} = \frac{20\times 6^2}{12} + \frac{40\times 6}{8} = 60 + 30 = 90\,kNm$$

$$-FEM_{BC} = FEM_{CB} = \frac{wl^2}{12} + \frac{Pl}{8} = \frac{20\times 6^2}{12} + \frac{60\times 6}{8} = 60 + 45 = 105\,kNm$$

$$-FEM_{CD} = FEM_{DC} = \frac{Pl}{8} = \frac{80\times 6}{8} = 60\,kNm$$

Joint	Member	K (stifness)	ΣK (Total stifness)	D.F
A	—	—	—	1
B	BA	$\frac{3EI}{6} = 0.5EI$	1.16EI	0.43
	BC	$4EI/6 = 0.66EI$		0.57
C	CB	$\frac{4EI}{6} = 0.66EI$	1.16EI	0.57
	CD	$3EI/6 = 0.5EI$		0.43
D	—	—	—	1

Cont..

A		B		C	D	
	0.429	0.571	0.571	0.429		Distribution factors
-90	+90	-105	+105	-60	+60	Fixed End Moments
+90	+45			-30	-60	Release A& D, and carry over
0	**+135**	**-105**	**+105**	**-90**	**0**	Initial moments
	-12.87	-17.13	-8.565	-6.435		Distribution
		-4.283	-8.565			Carry over
	+1.837	+2.445	4.89	3.674		Distribution
		+2.445	1.223			Carry over
	-1.049	-1.396	-0.698	-0.524		Distribution
0	**+122.92**	**-122.92**	**+93.29**	**- 93.29**	**0**	Final Moments

Example-5 : Analyse the beam shown in figure by moment distribution method and draw SFD and BMD. **(G.U., Dec. 2006)**

Solution :

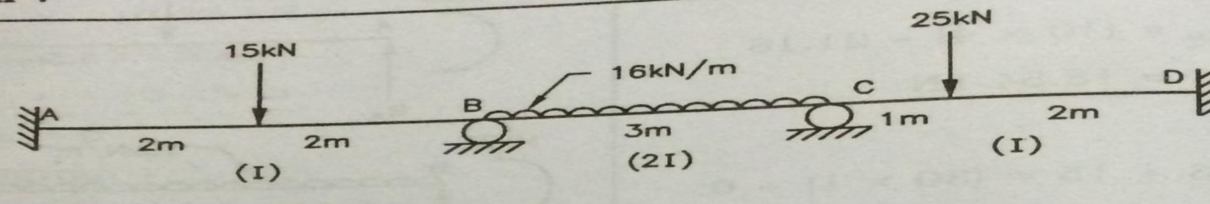


FIG. 3.21

(a) Fixed End Moments (FEM) :

$$M_f\ AB = -\frac{Wl}{8} = -\frac{15 \times 4}{8} = -7.5 \text{ kN.m}$$

$$M_f\ BA = +\frac{Wl}{8} = 7.5 \text{ kN.m}$$

$$M_f\ BC = -\frac{wl^2}{12} = -\frac{16 \times 3^2}{12} = -12 \text{ kN.m}$$

$$M_f\ CB = +\frac{wl^2}{12} = 12 \text{ kN.m}$$

$$M_f\ CD = -\frac{W\,ab^2}{l^2} = -\frac{25 \times 1 \times 2^2}{3^2} = -11.11 \text{ kN.m}$$

$$M_f\ DC = +\frac{W\,ba^2}{l^2} = \frac{25 \times 2 \times 1^2}{3^2} = 5.55 \text{ kN.m}$$

(b) Distribution Factors (D.F.) :

Sr.No.	Joint	Member	k	Σk	D.F. = $\frac{k}{\Sigma k}$
1.	B	BA	$\frac{4\,EI}{4} = 1.0\ EI$	3.67 EI	0.27
		BC	$\frac{4\,E\,(2I)}{3} = 2.67\ EI$		0.73
2.	C	CB	$\frac{4\,E\,(2I)}{3} = 2.67\ EI$	4.0 EI	0.67
		CD	$\frac{4\,EI}{3} = 1.33\ EI$		0.33
3.	A	–	–	–	0
4.	D	–	–	–	0

(c) Moment Distribution (M.D.) :

	A	B		C		D	
	0	0.27	0.73	0.67	0.33	0	D.F.
	−7.5	7.5	−12	12	−11.11	5.55	F.E.M.
Sum	0	1.21	3.28	−0.59	−0.30	0	Balance
	0.60	0	−0.30	1.64	0	−0.15	C.O.
	0	0.08	0.22	−1.10	−0.54	0	Balance
	0.04	0	−0.55	0.11	0	−0.27	C.O.
	0	0.15	0.40	−0.07	−0.04	0	Balance
	0.075	0	−0.035	0.20	0	−0.02	C.O.
	0	0.009	0.026	−0.13	−0.07	0	Balance
	−6.79	8.95	−8.95	12.06	−12.06	5.11	Final moments

balancing

$M_{AB} = -6.79$ kN.m.

$M_{BA} = 8.95$ kN.m, $M_{BC} = -8.95$ kN.m

$M_{CB} = 12.06$ kN.m, $M_{CD} = -12.06$ kN.m

$M_{DC} = 5.11$ kN.m

(d) B.M. diagram :

Simply supported moments,

$$\text{Span AB,}\quad M = \frac{Wl}{4} = \frac{15 \times 4}{4} = 15 \text{ kN.m}$$

$$\text{Span BC,}\quad M = \frac{wl^2}{8} = \frac{16 \times 3^2}{8} = 18 \text{ kN.m}$$

$$\text{Span CD,}\quad M = \frac{W\,ab}{l} = \frac{25 \times 1 \times 2}{3} = 16.67 \text{ kN.m}$$

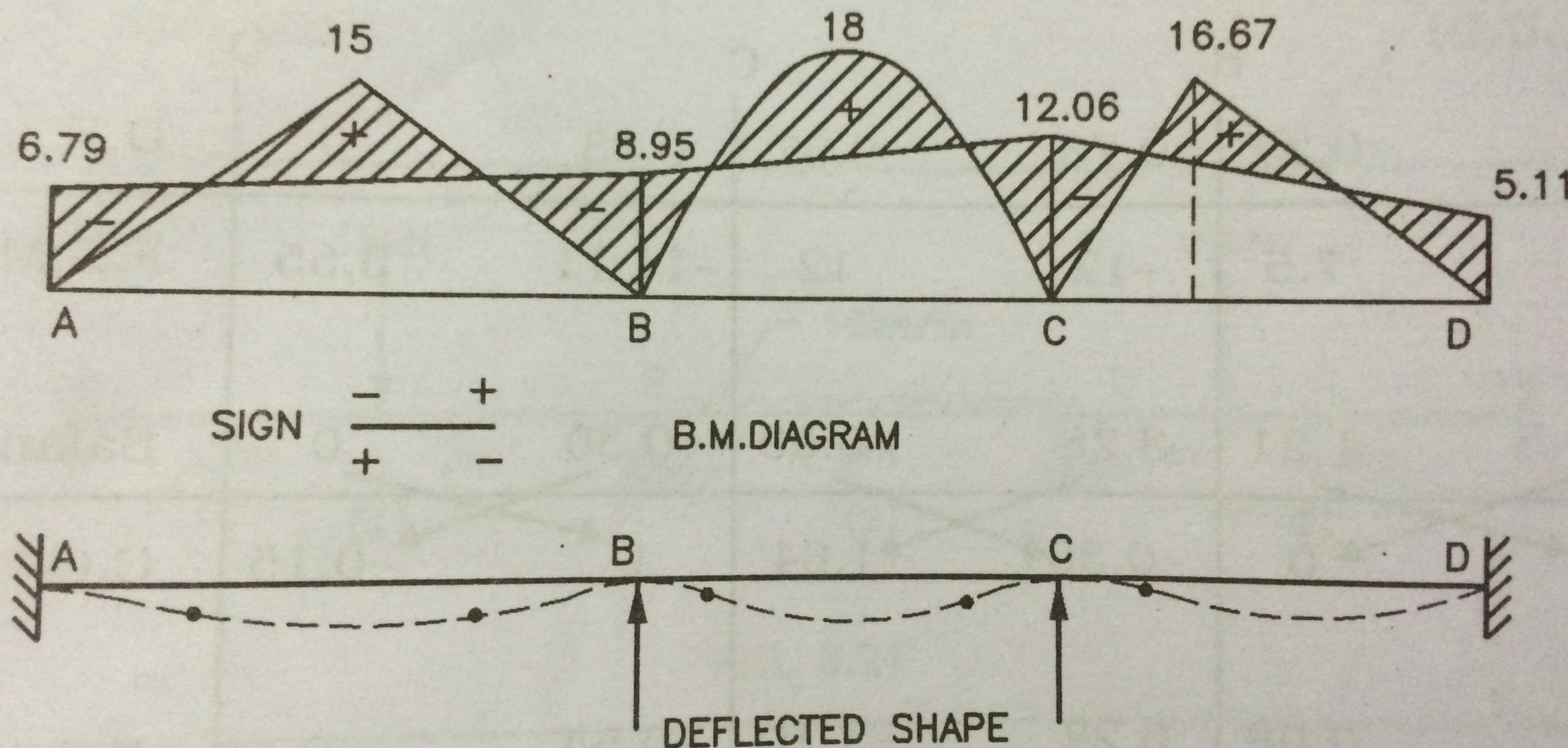
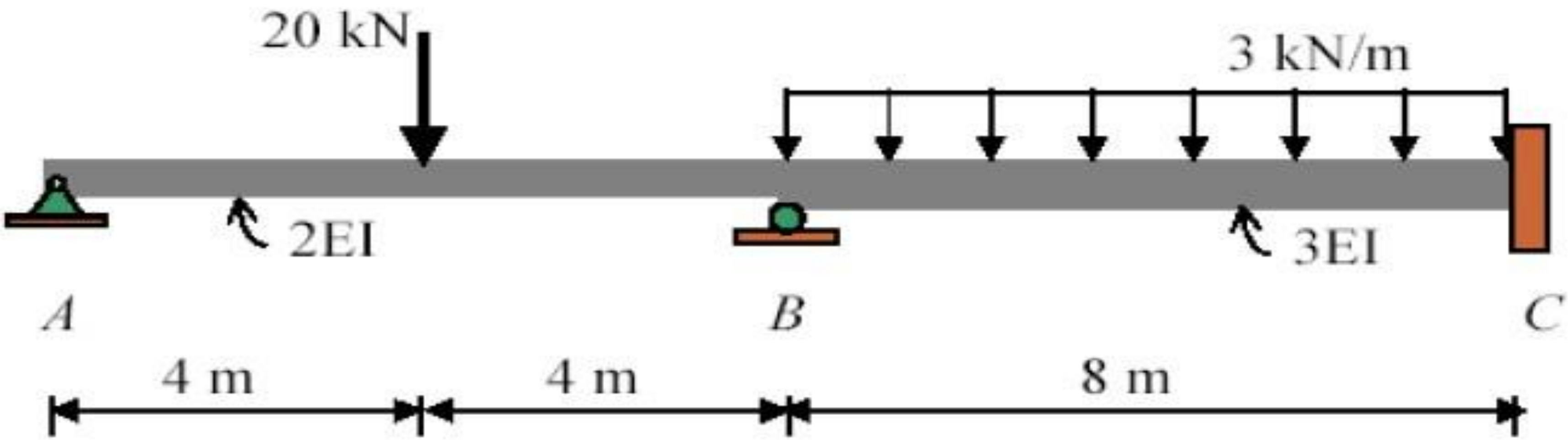

15
18
16.67
12.06
6.79
8.95
5.11
A
B
C
D
SIGN
B.M.DIAGRAM
A
B
C
D
DEFLECTED SHAPE

FIG. 3.22

DRCFBY HDCRFY

Support B settles by 10 mm. $E = 200GPa, \quad I = 50\times10^{6} mm^{4}$



$$DF_{BA} = \frac{3(2EI/8)}{3(2EI/8)+4(3EI/8)} = 0.333 \qquad DF_{BC} = \frac{4(3EI/8)}{3(2EI/8)+4(3EI/8)} = 0.6$$

$$\frac{R/}{8} \quad \frac{20XÚ}{8} = 20$$

$$\frac{›‹/^2}{12} \;-\!- \;\frac{3 \quad 8^2}{12} \;-\!- \quad k$$

$$FEM_{AB} = \frac{\cdot \; \cdot}{8} - \frac{EI\delta}{L^2}$$

2ft $\frac{6 \times 2 \times 200 \times 10" \times 30 \quad 10‘ \quad 10*'“ X 10 X 10*’}{2}$

$$- \;—20-18.73 \quad —38.75$$

$$= 20 - \frac{6 \times 2 \times 200 \times 10’ \times 50 \times 10^6 \times 10^{*12} \quad 10 \times 10*’}{8^2}$$

$$- 20 - 18.75 - 1.25\ kNm$$

$$_{BC} = \frac{\quad}{12} \cdot \frac{\quad}{L^2}$$

$$= -16 + \frac{6 \times 3 \times 200 \times 10^6 \times 50 \times 10^6 \times 10^{-12} \times 10 \times 10^{-3}}{g2}$$

=−1ö+28125=ł2ł25ÆVœ

$$\frac{wl^2}{12} \quad \frac{6EI\delta}{L}$$

$$\$ \quad \frac{6 \text{ x } 3 \text{ x } 200 \text{ x } 0^6 \text{ x } 50 \text{ x } 0^6 \text{ x } 0^{*12} \text{ x } 10 \text{ x } 10^{*\prime}}{\quad}$$

= 1ò + 28.125 = 44.123

A	B		C	
1	0.333	0.667	0	
-38.75	+1.25	12.125	44.125	Fixed End Moments
38.75	19.375			Release A, and carry over
0.0	20.625	12.125	44.125	M∘ *
	-10.906	-21 844	0	„stribution
			-10.922	∘*∧
0.0	+ 9.719	-9.719	+33.203	Final Moments

Thank you!
STABIL