# UNIT - IV

# MATRIX METHOD OF ANALYSIS

## THE DIRECT STIFFNESS METHOD

### INTRODUCTION

All known methods of structural analysis are classified into two distinct groups:-

1. force method of analysisand

2. displacement method ofanalysis.

In module 2, the force method of analysis or the method of consistent deformation is discussed. An introduction to the displacement method of analysis is given in module 3, where in slope-deflection method and moment- distribution method are discussed. In this module the direct stiffness method is discussed. In the displacement method of analysis the equilibrium equations are written by expressing the unknown joint displacements in terms of loads by using load- displacement relations. The unknown joint displacements (the degrees of freedom of the structure) are calculated by solving equilibrium equations. The slope-deflection and moment-distribution methods were extensively used before the high speed computing era. After the revolution in computer industry, only direct stiffness method isused.

The displacement method follows essentially the same steps for both statically determinate and indeterminate structures. In displacement /stiffness method of analysis, once the structural model is defined, the unknowns (joint rotations and translations) are automatically chosen unlike the force method of analysis. Hence, displacement method of analysis is preferred to computer implementation. The method follows a rather a set procedure. The direct stiffness method is closely related to slope-deflectionequations.

The general method of analyzing indeterminate structures by displacement method may be traced to Navier (1785-1836). For example consider a four member truss as shown in Fig.23.1.The given truss is statically indeterminate to second degree as there are four bar forces but we have only two equations of equilibrium. Denote each member by a number, for example (1), (2), (3) and (4).

Let $\alpha_i$be the angle, the $i$-th member makes with the horizontal. Under the action of external loads $P_x$ and $P_y$, the joint $E$ displaces to $E'$. Let $u$ and $v$ be its vertical and horizontal displacements. Navier solved this problem as follows.

In the displacement method of analysis $u$ and $v$ are the only two unknowns for this structure. The elongation of individual truss members can be expressed in terms of these two unknown joint displacements. Next, calculate bar forces in the members by using force–displacement relation.
The unknown displacements may be calculated by solving the equilibrium equations. In displacement method of analysis, there will be exactly as many equilibrium equations as there are unknowns.

Let an elastic body is acted by a force F and the corresponding displacement be $u$ in the direction of force. In module 1, we have discussed force- displacementrelationship. The force ($F$) is related to the displacement ($u$) for the linear elastic material by therelation

$$F = ku \tag{23.1}$$

where the constant of proportionality $k$ is defined as the stiffness of the structure and it has units of force per unit elongation. The above equation may also be written as

$$u = aF \tag{23.2}$$

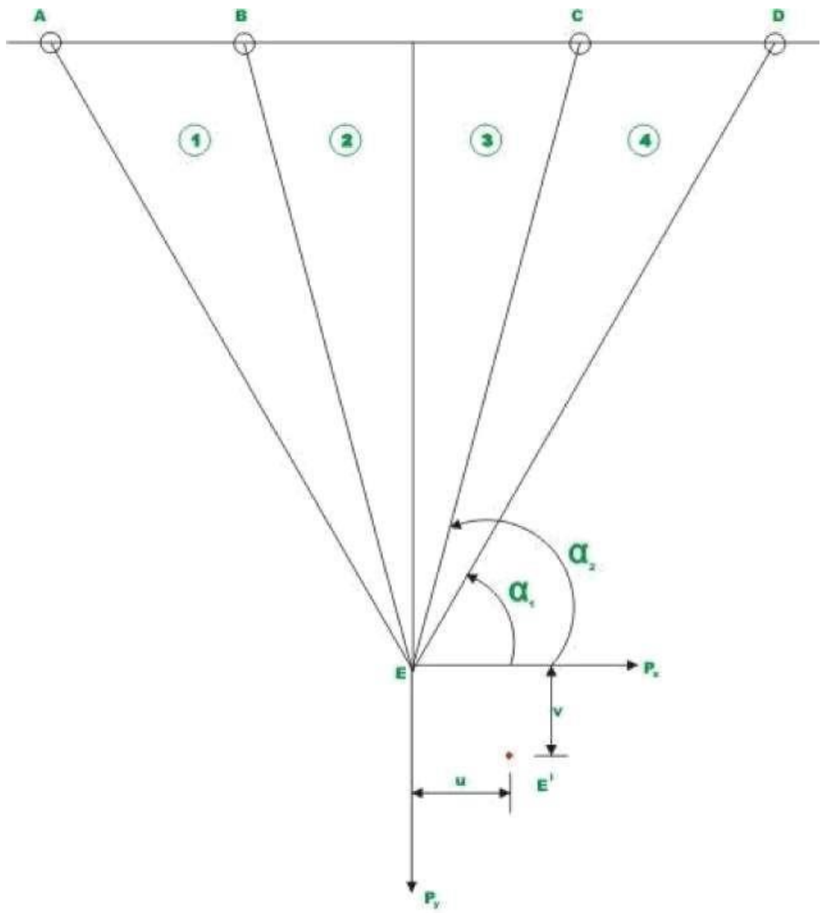



**Fig. 23.1 Four member truss**

The constant $a$is known as flexibility of the structure and it has a unit of displacement per unit force. In general the structures are subjected to $n$ forces at $n$ different locations on the structure. In such a case, to relate displacement at$i$to load at $j$, it is required to use flexibility coefficients with subscripts. Thus theflexibility coefficient $a_{ij}$is the deflection at $i$due to unit value of force applied at $j$ . Similarly the stiffness coefficient$k_{ij}$is defined as the force generated at$i$

due to unit displacement at $j$ with all other displacements kept at zero. Toillustrate this definition, consider a cantilever beam which is loaded as shown in Fig.23.2. The two degrees of freedom for this problem are vertical displacementat $B$ and

rotation at $B$. Let them be denoted by and $(=\theta_1)$. Denote thevertical force P byand the tip moment $M$ by. Now apply a unit vertical force along calculate deflection and . The vertical deflection is denoted by flexibility coefficient and rotation is denoted by flexibilitycoefficient . Similarly, by applying a unit force along , one could calculate flexibility coefficient and . Thus is the deflection at 1 corresponding to due to unit force applied at 2 in the direction of By using the principle of superposition, the displacements and are expressed as the sum of displacements due to loads andacting separately on the beam. Thus,

The above equation may be written in matrix notation as

$$\{u\}=[a]\{P\}$$

where, $\{u\}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$; $\{a\}=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; and $\{P\}=\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$

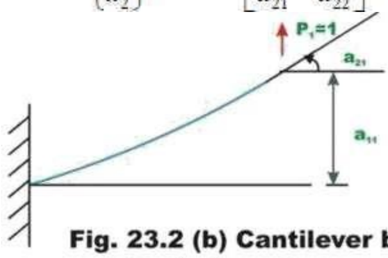



**Fig. 23.2 (b) Cantilever beam with unit load along $P_1$**


**Fig.23.2(a) Cantilever beam**

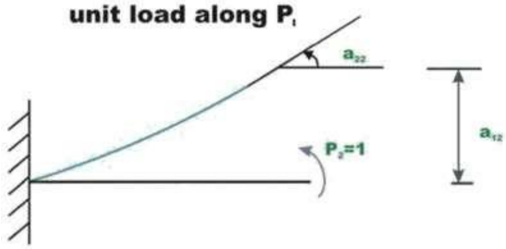



**Fig. 23.2 © Cantilever beam with unit moment along $P_2$**

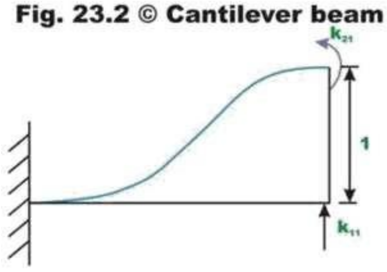



**Fig. 23.2(d) Cantilever beam with unit displacement along $u_1$**

The forces can also be related to displacements using stiffness coefficients. Apply a unit displacement along (see Fig.23.2d) keeping displacement as zero.

Calculate the required forces for this case as $k_{11}$and $k_{21}$ .Here, $k_{21}$ represents force developed along $P_2$ when a unit displacement along is introduced keeping=0. Apply a unit rotation along(vide Fig.23.2c),keeping□ 0

Calculate the required forces for this configuration $k_{12}$and $k_{22}$. Invoking the principle of superposition, the forces $P_1$ and $P_2$ are expressed as the sum of

forces developed due to displacements and acting separately on the beam. Thus,

$$\{P\}=[k]\{u\}$$

$$\text{where, } \{P\}=\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix};\ \{k\}=\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix};\text{and } \{u\}=\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}.$$

$k$is defined as the stiffness matrix of the beam.

In this lesson, using stiffness method a few problems will be solved. However this approach is very rudimentary and is suited for hand computation. A more formal approach of the stiffness method will be presented in the next lesson.

## A SIMPLE EXAMPLE WITH ONE DEGREE OF FREEDOM

Consider a fixed–simply supported beam of constant flexural rigidity $EI$ and span $L$ which is carrying a uniformly distributed load of w kN/m as shown in Fig.23.3a. If the axial deformation is neglected, then this beam is kinematically indeterminate to first degree. The only unknown joint displacement is $\theta_B$.Thus the degrees of freedom for this structure is one (for a brief discussion on degrees of freedom, please see introduction to module 3).The analysis of above structure by stiffness method is accomplished in following steps:

Recall that in the flexibility /force method the redundants are released (i.e. made zero) to obtain a statically determinate structure. A similar operation in the stiffness method is to make all the unknown displacements equal to zero by altering the boundary conditions. Such an altered structure is known as kinematically determinate structure as all joint displacements are known in this case. In the present case the restrained structureisobtained by preventing the rotation at B as shown in Fig.23.3b. Apply all the external loads on the kinematically determinate structure. Duetorestraint at $B$, a moment $M_B$ is developed at $B$. In the stiffness method weadopt the following sign convention. Counterclockwise moments and counterclockwise rotations are taken as positive, upward forces and displacements are taken as positive.


Fig.23.3(a) Cantilever beam

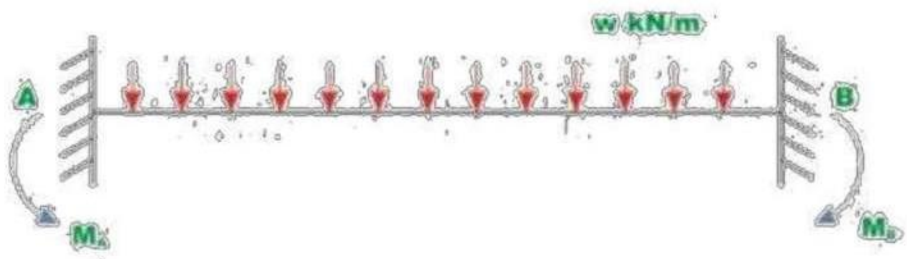



Fig. 23.3b Kinematically determinate beam

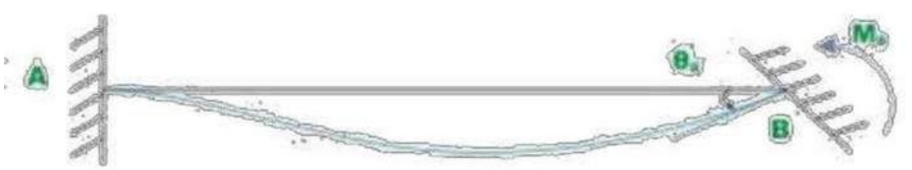



Fig. 23.3 ©

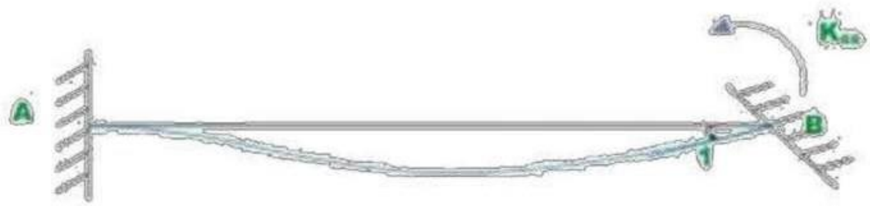



Fig. 23.3 (d) Computation of stiffness co-efficients

## TWO DEGREES OF FREEDOM STRUCTURE

Consider a plane truss as shown in Fig.23.4a.There is four members in the truss and they meet at the common point at *E*. The truss is subjected to external loadsandacting at *E*. In the analysis, neglect the self weight of members. Thereare two unknown displacements at joint *E* and are denoted byand.Thus the structure is kinematically indeterminate to second degree. The applied forces and unknown joint displacements are shown in the positive directions. The members are numbered from (1), (2), (3) and (4) as shown in the figure. The length and

axial rigidity of $i$-th member is $l_i$and $EA_i$respectively. Now it is soughtto evaluate andby stiffness method. This is done in following steps:

In the first step, make all the unknown displacements equal to zero by altering the boundary conditions as shown in Fig.23.4b. On thisrestrained/kinematically determinate structure, apply all the external loads except the joint loads and calculate the reactions corresponding to unknown joint displacements and. Since, in the present case, there are no external loads other than the joint loads, the reactions $(R_L)_1$and $(R_L)_2$ will be equal to zero. Thus,

In the next step, calculate stiffness coefficients $k_{11}$, $k_{21}$, $k_{12}$ and$k_{22}$ .This is done as follows. First give a unit displacement along $u_1$ holdingdisplacement along to zero and calculate reactions at *E* correspondingto unknown displacements and in thekinematically determinate structure. They are denoted by $k_{11}$,$k_{21}$. The joint stiffness $k_{11}$,$k_{21}$of thewhole truss is composed of individual member stiffness of the truss. This isshowninFig.23.4c.Nowconsiderthemember$AE$.Undertheactionofunit displacement along, the joint *E* displaces to. Obviously the new length is not equal to length *AE*. Let us denote the new length of themembersby$l_1 + \Delta l_1$, where $l$, is the change in length of the member AE′. The member $AE$ ′also makes an angle with the horizontal. This isjustified as $l_1$ is small. From the geometry, the change in length of the members $AE'$is

(23.11a)

The elongation $\Delta l_1$ is related to the force in the member $AE'$, $F_{AE'}$ by

$$\Delta l_1 = \frac{F_{AE'} l_1}{A_1 E} \qquad (23.11b)$$

Thus from (23.11a) and (23.11b), the force in the members $AE$□ is

$$F'_{AE} = \frac{EA_1}{l_1} \cos\theta_1 \qquad (23.11c)$$

This force acts along the member axis. This force may be resolved alongand directions . Thus the horizontal component of force

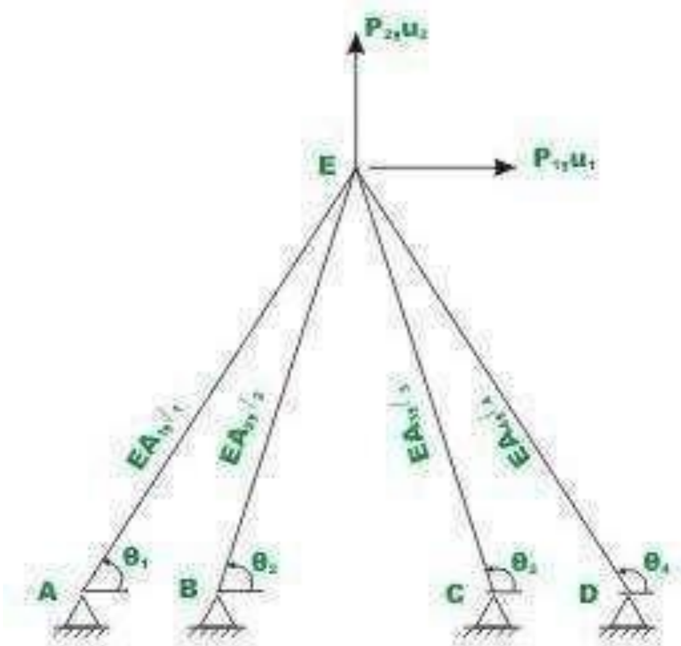



**Fig 23.4a A four - member truss**

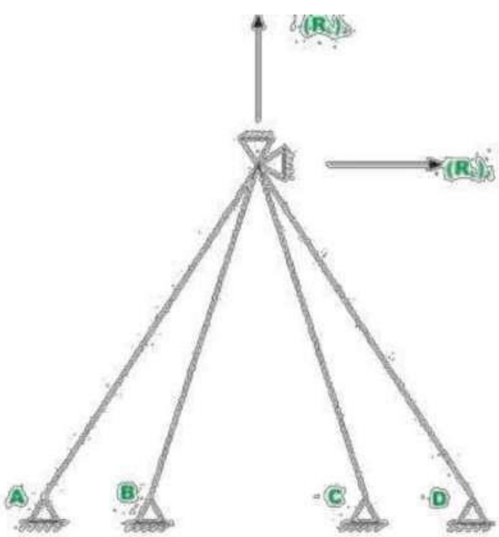



**Fig. 23.4b Kinematically determinate structure**

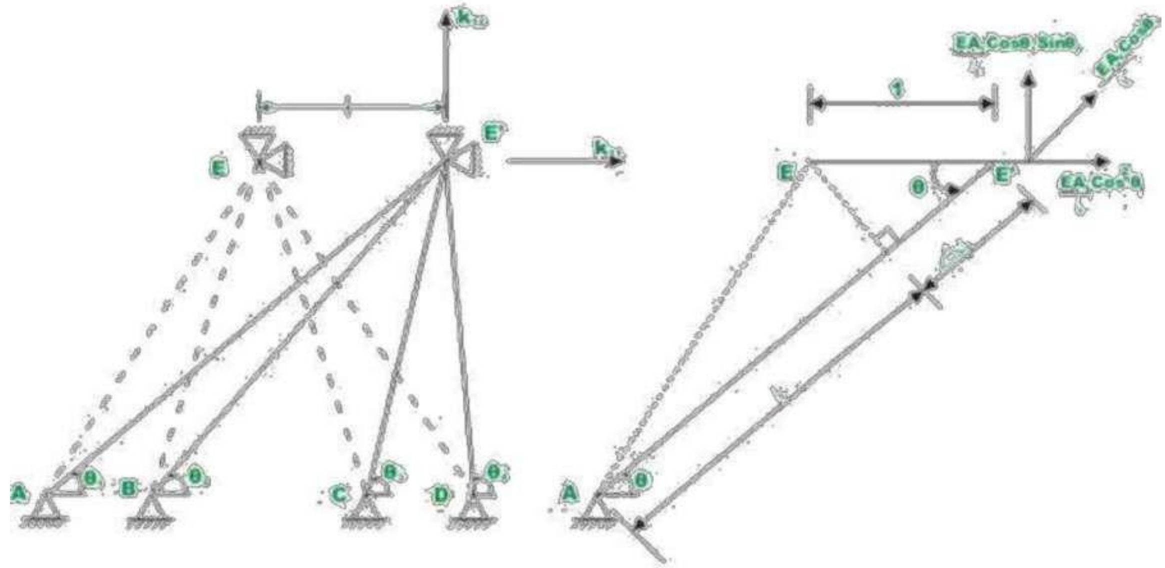
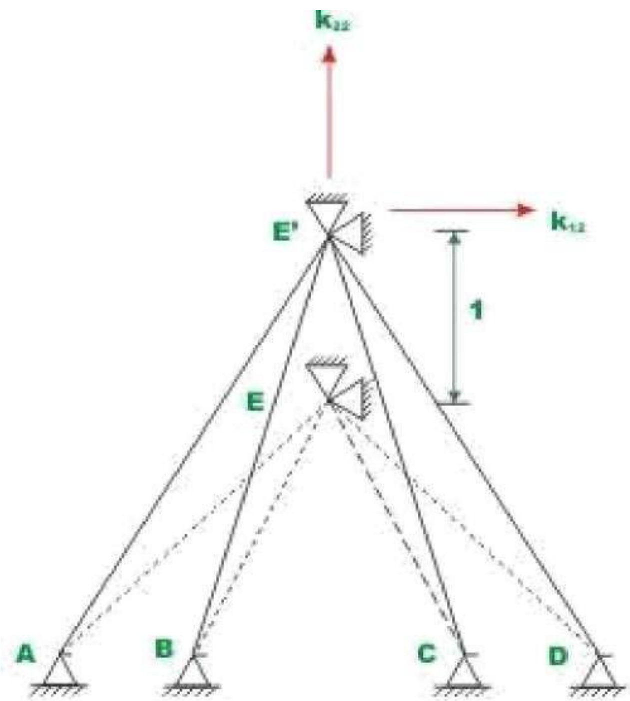



**Fig. 23.4c Unit displacement along u$_1$**



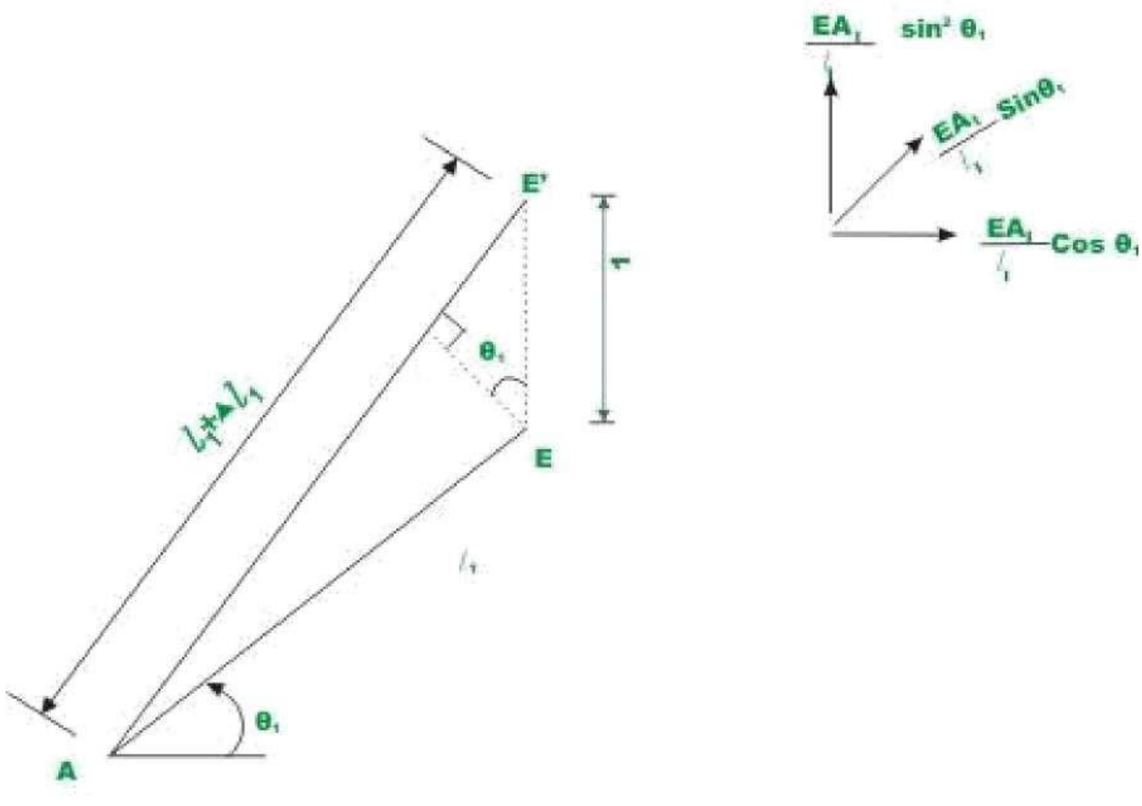




**Fig.23.4d Unit displacement along $u_2$**

Expressions of similar form as above may be obtained for all members. The sum of all horizontal components of individual forces gives us the stiffness coefficient

$k_{11}$ and sum of all vertical component of forces give us the required stiffness coefficient $k_{21}$

$$k_{11} = \frac{EA_1}{l_1}\cos^2\theta_1 + \frac{EA_2}{l_2}\cos^2\theta_2 + \frac{EA_3}{l_3}\cos^2\theta_3 + \frac{EA_4}{l_4}\cos^2\theta_4$$

$$k_{11} = \sum_{i=1}^{4} \frac{EA_i}{l_i}\cos^2\theta_i \tag{23.15}$$

Similarly, $k_{12} = \sum_{i=1}^{4} \frac{EA_i}{l_i}\sin\theta_i \cos\theta_i$ (23.16)

B. Joint forces in the original structure corresponding to unknown displacements $u_1$ and $u_2$ are

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} \tag{23.17}$$

Now the equilibrium equations at joint $E$ states that the forces in the original structure are equal to the superposition of (i) reactions in the kinematically restrained structure corresponding to unknown joint displacements and (ii) reactions in the restrained structure due to unknown displacements themselves. This may be expressed as,

$$\{F\}=\begin{Bmatrix}F_1\\F_2\end{Bmatrix};$$

$$\{R_L\}=\begin{Bmatrix}(R_L)_1\\(R_L)_2\end{Bmatrix}$$

$$[k]=\begin{bmatrix}k_{11} & k_{12}\\k_{21} & k_{22}\end{bmatrix}$$

$$\{u\}=\begin{Bmatrix}u_1\\u_2\end{Bmatrix} \tag{23.20}$$

For example take $P_1=P_2=P$, $L_i=\dfrac{L}{\sin\theta_i}$, $A_1=A_2=A_3=A_4=A$ and $\theta_1=35°$, $\theta_2=70°$, $\theta_3=105°$ and $\theta_4=140°$

Then.

$$\{F\}=\begin{Bmatrix}P\\P\end{Bmatrix} \tag{23.21}$$

$$\{R_L\}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

$$k_{11}=\sum\frac{EA}{L}\cos^2\theta_i\,\sin\theta_i=0.9367\frac{EA}{L}$$

$$k_{12}=\sum\frac{EA}{L}\sin^2\theta_i\,\cos\theta_i=0.0135\frac{EA}{L}$$

$$k_{21} = \sum \frac{EA}{L} \sin^2 \theta_i \cos \theta_i = 0.0135 \frac{EA}{L}$$

$$k_{22} = \sum \frac{EA}{L} \sin^3 \theta_i = 2.1853 \frac{EA}{L} \qquad (23.22)$$

$$\begin{Bmatrix} P \\ P \end{Bmatrix} = \frac{EA}{L} \begin{bmatrix} 0.9367 & 0.0135 \\ 0.0135 & 2.1853 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Solving which, yields

$$u_1 = 1.0611 \frac{L}{EA}$$

$$u_2 = 0.451 \frac{L}{EA}$$

## Example

Analyze the plane frame shown in Fig.23.5a by the direct stiffness method. Assume that the flexural rigidity for all members is the same. Neglect axial displacements.

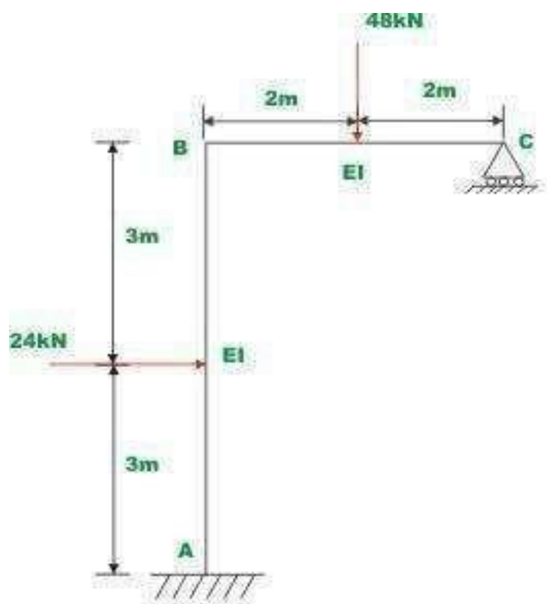



**Fig 23.5a Plane - frame of Example 23.1**

## Solution

In the first step identify the degrees of freedom of the frame .The given frame has three degrees of freedom (see Fig.23.5b):

B. Two rotations as indicated by $u_1$ and $u_2$and

C. One horizontal displacement of joint B and C as indicated by $u_3$ .

In the next step make all the displacements equal to zero by fixing joints B and C as shown in Fig.23.5c. On this kinematically determinate structure apply all the external loads and calculate reactions corresponding to unknown joint displacements .Thus,

$$\left(R_D^F\right)_1 = \frac{48\times2\times4}{16} + \left(-\frac{24\times3\times9}{36}\right)$$

(1)

$$= 24 - 18 = 6 \text{ kN.m}$$

$$\left(R_D^F\right)_2 = -24 \text{ kN.m}$$

$$\left(R_D^F\right)_3 = 12 \text{ kN.m} \tag{2}$$

Thus,

$$\begin{Bmatrix} \left(R_D^F\right)_1 \\ \left(R_D^F\right)_2 \\ \left(R_D^F\right)_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} \tag{3}$$

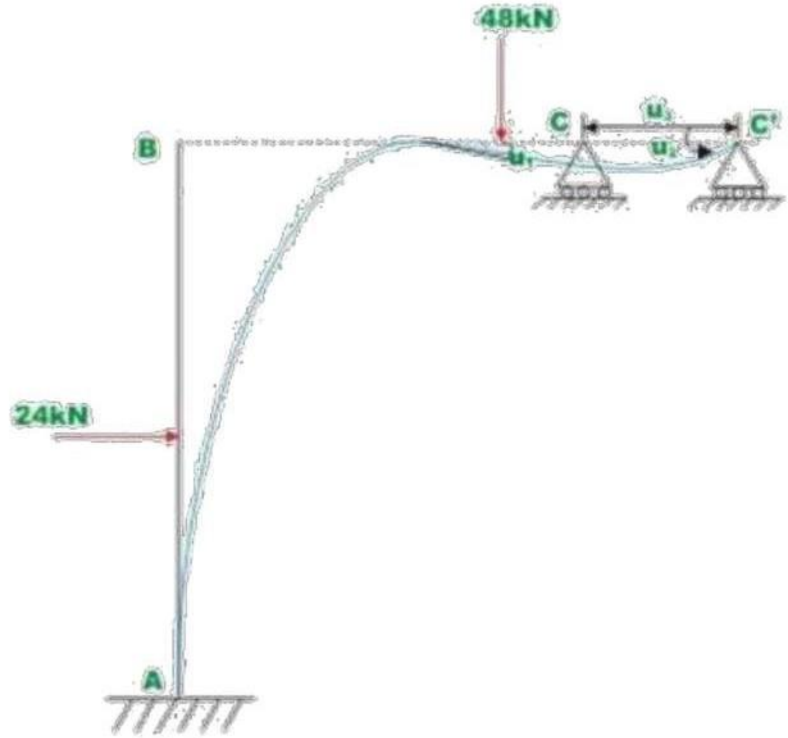



Fig 23.5b Approximate deflected shape

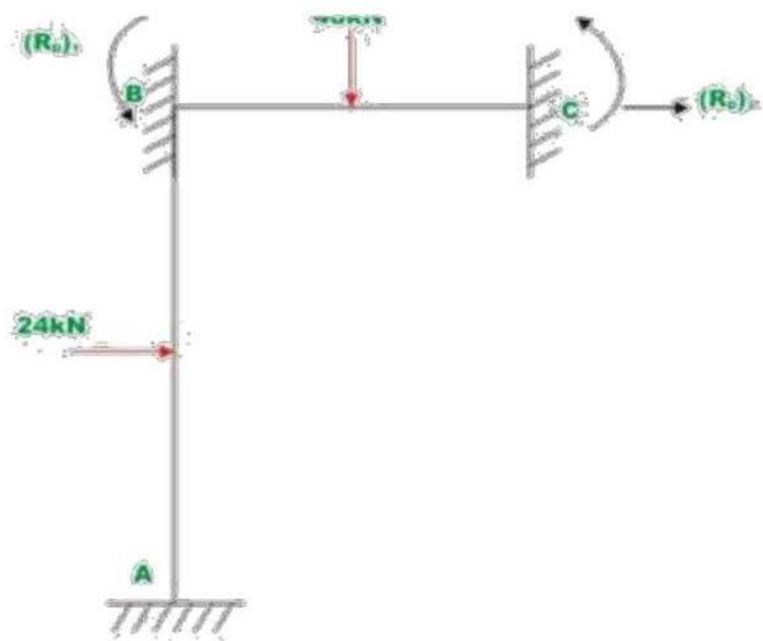



Fig 23.5c Kinematically restrained structure

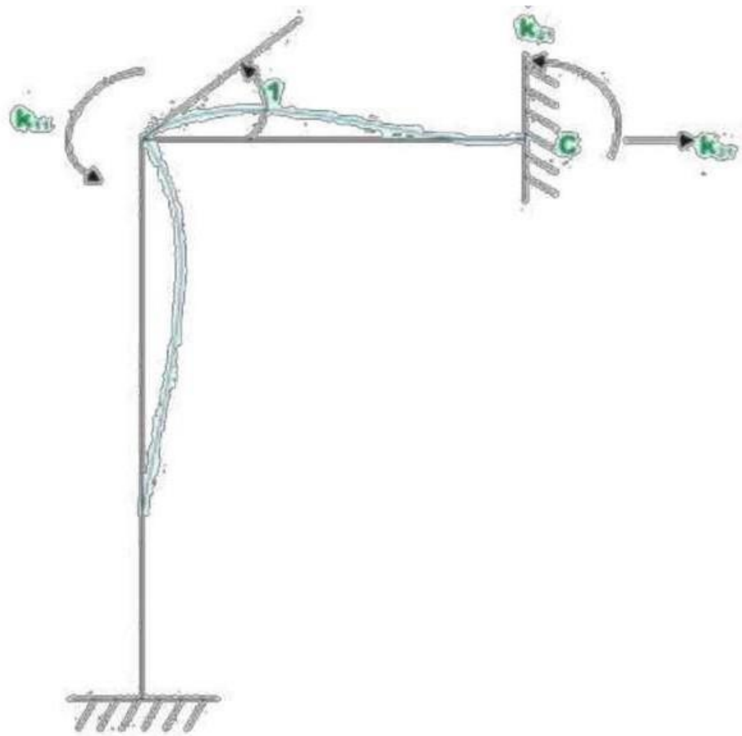


Fig.23.5d Unit displacement along $u_1$

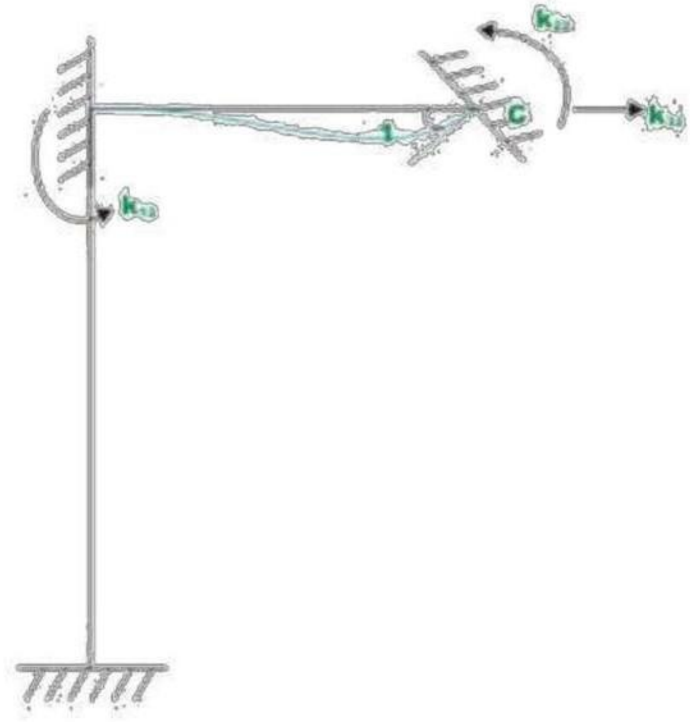


Fig 23.5e Unit displacement along $u_2$

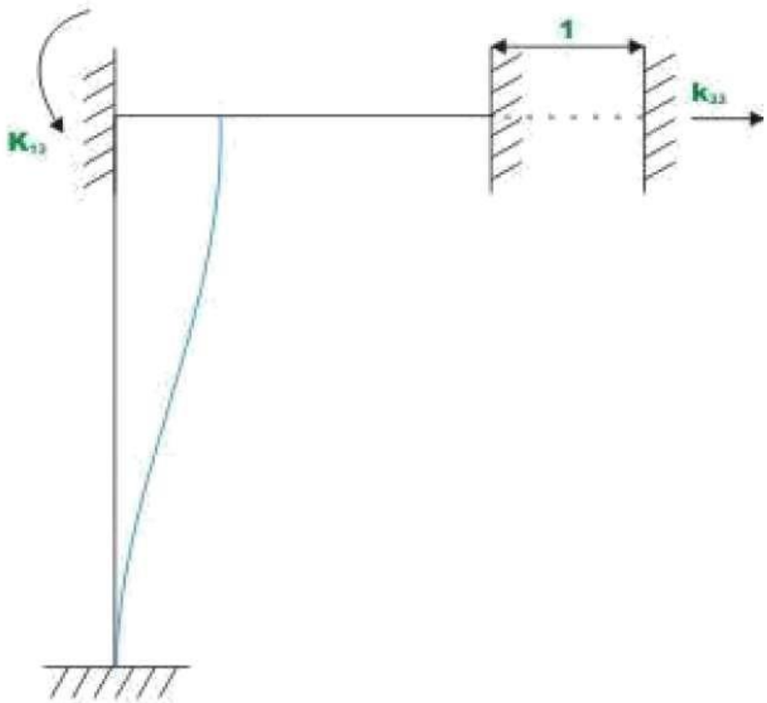



**Fig. 23.5f Unit displacement along $u_3$**

$$k_{11} = \frac{4EI}{4} + \frac{4EI}{6} = 1.667$$

$$k_{21} = \frac{2EI}{4} = 0.5EI$$

$$k_{31} = -\frac{6EI}{6 \times 6} = -0.166EI \qquad (4)$$

Similarly, apply a unit rotation along $u_2$and calculate reactions corresponding to three degrees of freedom (see Fig.23.5e)

$$k_{12} = 0.5EI$$

$$k_{22} = EI$$

$$k_{32} = 0 \qquad (5)$$

Apply a unit displacement along $u_3$and calculate joint reactions corresponding to unknown displacements in the kinematically determinate structure.

$$k_{13} = -\frac{6EI}{L^2} = -0.166E$$

$$k_{23} = 0$$

$$k_{33} = \frac{12EI}{6^3} = 0.056EI \qquad (6)$$

Finally applying the principle of superposition of joint forces, yields

Now,
$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -24 \\ 12 \end{Bmatrix} + EI \begin{bmatrix} 1.667 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ as there are no loads applied along $u_1$,$u_2$ and $u_3$.Thus the

unknown displacements are,

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = -\frac{1}{EI} \begin{bmatrix} 1 & 0.5 & -0.166 \\ 0.5 & 1 & 0 \\ -0.166 & 0 & 0.056 \end{bmatrix}^{-1} \begin{Bmatrix} 6 \\ -24 \\ -24 \end{Bmatrix} \qquad (7)$$

Solving

$$u_1 = \frac{18.996}{EI}$$

$$u_2 = \frac{14.502}{EI}$$

$$u_3 = -\frac{270.587}{EI} \qquad (8)$$

## Example 24.1

Analyse the two member truss shown in Fig. 24.12a. Assume $EA$ to be constant for all members. The length of each member is $5m$.

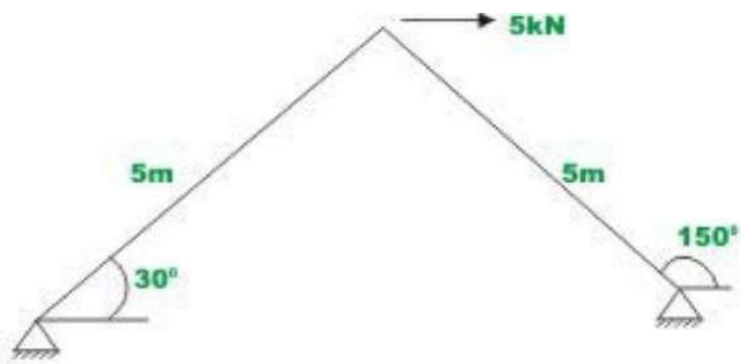



**Fig 24.12(a) Example 24.1**

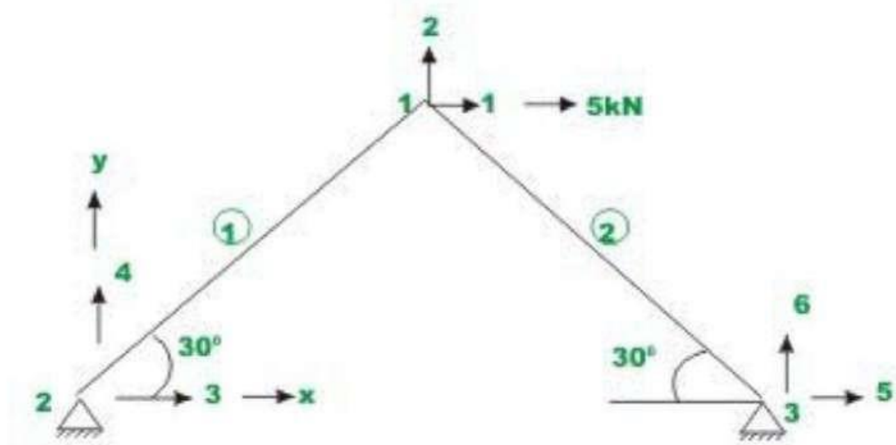



**Fig 24.12(b) Members and node numbering**

The co-ordinate axes, the number of nodes and members are shownin

Fig.24.12b. The degrees of freedom at each node are also shown. By inspection it is clear that the displacement $u_3=u_4=u_5=u_6=0$. Also the external loads are

$$p_1=5 \quad \text{kN} \quad ; p_2=0 \quad \text{kN}. \tag{1}$$

Now member stiffness matrix for each member in global co-ordinate system is $(\theta_1=30°)$.

$$[k^1]=\frac{EA}{5}\begin{bmatrix} 0.75 & 0.433 & -0.75 & -0.433 \\ 0.433 & 0.25 & -0.433 & -0.25 \\ -0.75 & -0.433 & 0.75 & 0.433 \\ -0.433 & -0.25 & 0.433 & 0.25 \end{bmatrix} \tag{2}$$

$$[k^2]=\frac{EA}{5}\begin{bmatrix} 0.75 & -0.433 & -0.75 & 0.433 \\ -0.433 & 0.25 & 0.433 & -0.25 \\ -0.75 & 0.433 & 0.75 & -0.433 \\ 0.433 & -0.25 & -0.433 & 0.25 \end{bmatrix} \tag{3}$$

The global stiffness matrix of the truss can be obtained by assembling the two member stiffness matrices. Thus,

$$[K]=\frac{EA}{5}\left[\begin{array}{cc|cccc} 1.5 & 0 & -0.75 & -0.433 & -0.75 & 0.433 \\ 0 & 0.5 & -0.433 & -0.25 & 0.433 & -0.25 \\ \hline -0.75 & -0.433 & 0.75 & 0.433 & 0 & 0 \\ -0.433 & -0.25 & 0.433 & 0.25 & 0 & 0 \\ -0.75 & 0.433 & 0 & 0 & 0.75 & -0.433 \\ 0.433 & -0.25 & 0 & 0 & -0.433 & 0.25 \end{array}\right] \tag{4}$$

Again stiffness matrix for the unconstrained degrees of freedom is,

$$[K] = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix} \quad (5)$$

Writing the load displacement-relation for the truss for the unconstrained degrees of freedom

$$\{p_k\} = [k_{11}]\{u_u\} \quad (6)$$

$$\begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\begin{Bmatrix} 5 \\ 0 \end{Bmatrix} = \frac{EA}{5}\begin{bmatrix} 1.5 & 0 \\ 0 & 0.5 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$u_1 = \frac{16.667}{EA} \quad ; u_2 = 0$$

Support reactions are evaluated using equation (24.30).

$$\{p_u\} = [k_{21}]\{u_u\}$$

Substituting appropriate values in equation (9),

$$\{p_u\} = \frac{EA}{5}\begin{bmatrix} -0.75 & -0.433 \\ -0.433 & -0.25 \\ -0.75 & 0.433 \\ 0.433 & -0.25 \end{bmatrix}\frac{1}{AE}\begin{Bmatrix} 16.667 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{pmatrix} -2.5 \\ -1.443 \\ -2.5 \\ 1.443 \end{pmatrix}$$

by equilibrium of joint

1. Also, $p_3+p_5+5=0$

Now force in each member is calculated as follows,

**Member 1**: $l=0.866$ ; $m=0.5$ ; $L=5m$.

$$\{p'\}=[k']\{u'\}$$

$$=[k'][T]\{u\} \quad (11)$$

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}[-0.866]\left\{\frac{16.667}{AE}\right\} = -2.88 \quad \text{kN}$$

**Member 2:** $l=-0.866$ ; $m=0.5$ ; $L=5\,m$ .

$$\begin{Bmatrix} p'_1 \\ p'_2 \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}\begin{Bmatrix} u_5 \\ v_6 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{L}\begin{bmatrix} l & m & -l & -m \end{bmatrix}\begin{Bmatrix} u_3 \\ v_4 \\ u_1 \\ v_2 \end{Bmatrix}$$

$$\{p'_1\} = \frac{AE}{5}\begin{bmatrix} -0.866 \end{bmatrix}\left\{\frac{16.667}{AE}\right\} = -2.88 \quad \text{kN}$$

\

## DIRECT STIFFNESS METHOD: BEAMS

In the earlier section, a few problems were solved using stiffness method from fundamentals. The procedure adopted therein is not suitable for computer implementation. In fact the load displacement relation for the entire structure was derived from fundamentals. This procedure runs into trouble when the structure is large and complex. However this can be much simplified provided we follow the procedure adopted for trusses. In the case of truss, the stiffness matrix of the entire truss was obtained by assembling the member stiffness matrices of individualmembers.

In a similar way, one could obtain the global stiffness matrix of a continuous beam from assembling member stiffness matrix of individual beam elements. Towards this end, we break the given beam into a number of beam elements. The stiffness matrix of each individual beam element can be written very easily. For example, consider a continuous beam $ABCD$ as shown in Fig. 27.1a. The given continuous beam is divided into three beam elements as shown inFig.

27.1b. It is noticed that, in this case, nodes are located at the supports. Thus each span is treated as an individual beam. However sometimes it is required to consider a node between support points. This is done whenever the cross sectional area changes suddenly or if it is required to calculate vertical or rotational displacements at an intermediate point. Such a division is shown inFig.

27.1c. If the axial deformations are neglected then each node of the beam will have two degrees of freedom: a vertical displacement (corresponding to shear) and a rotation (corresponding to bending moment). In Fig. 27.1b, numbers enclosed in a circle represents beam numbers. The beam $ABCD$ is divided into three beam members. Hence, there are four nodes and eight degrees of freedom. The possible displacement degrees of freedom of the beam are also shown in the figure. Let us use lower numbers to denote unknown degrees of freedom (unconstrained degrees of freedom) and higher numbers to denote known (constrained) degrees of freedom. Such a method of identification is adopted in this course for the ease of imposing boundary conditions directly on the structure stiffness matrix. However, one could number sequentially as shown in Fig. 27.1d. This is preferred while solving the problem on acomputer.

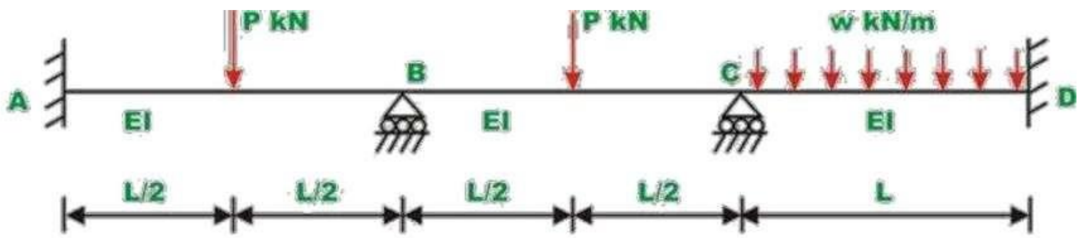



**Fig 27.1a Continuous beam**

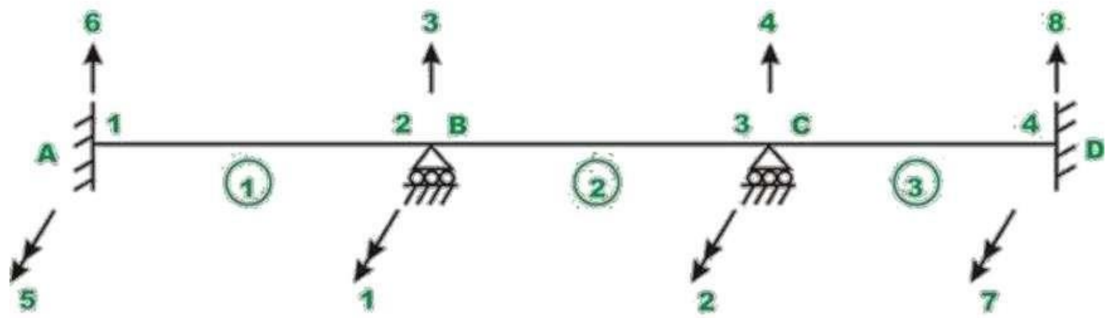



**Fig. 27.1b Member and node numbering**

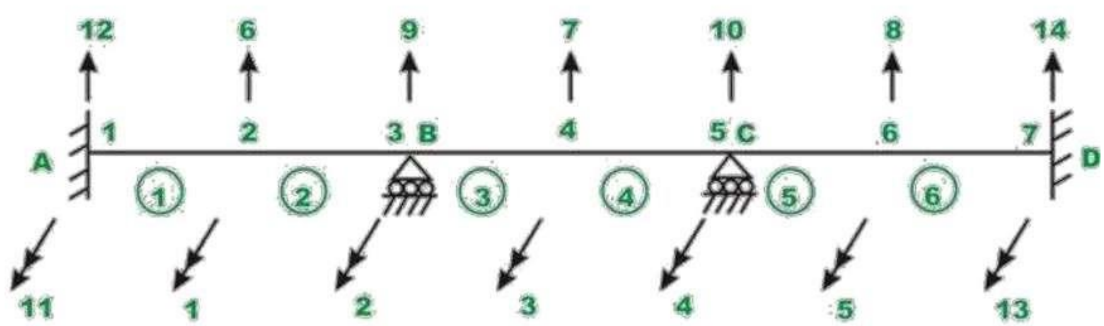



**Fig. 27.1c Member and node numbering**

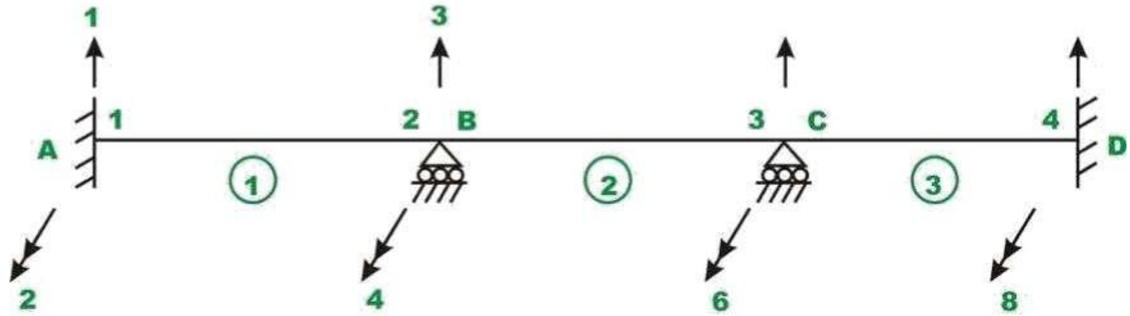



**Fig 27.1d Member and node numbering**

In the above figures, single headed arrows are used to indicate translational and double headed arrows are used to indicate rotational degrees of freedom.

## BEAM STIFFNESS MATRIX.

Fig. 27.2 shows a prismatic beam of a constant cross section that is fully restrained at ends in local orthogonal co-ordinate system $x'y'z'$ The beam ends

are denoted by nodes $j$ and $k$ The $x'$ axis coincides with the centroidal axis of the member with the positive sense being defined from $j$ to $k$ Let $L$ be the length of the member, $A$area of cross section of the member and $I_{zz}$is the moment of inertia about $z'$axis.

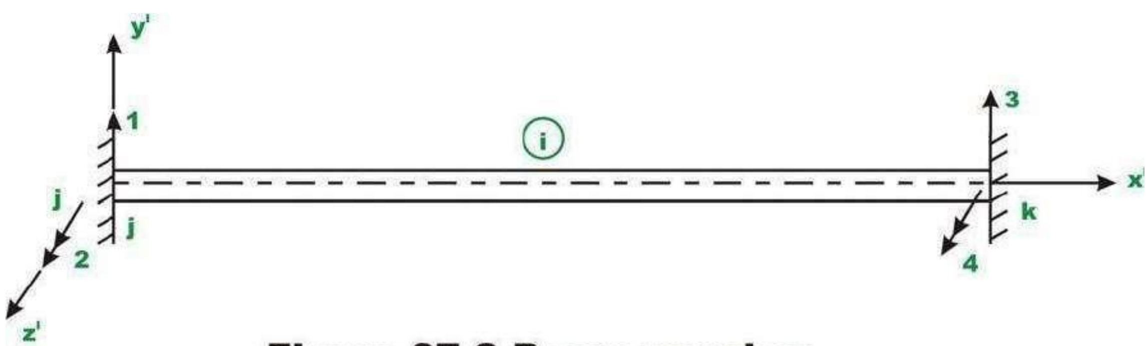
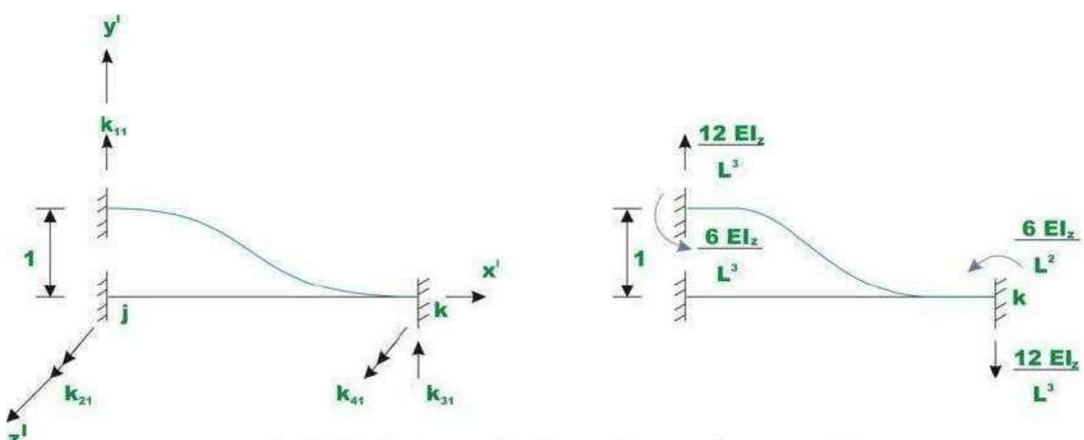


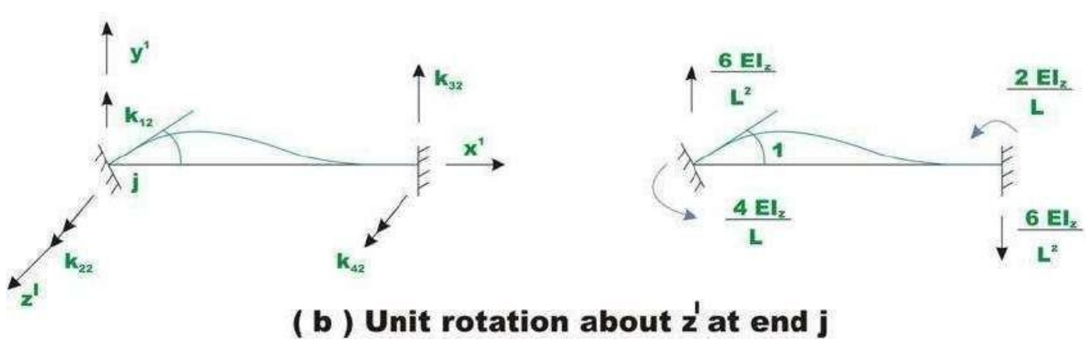

**Figure 27.2 Beam member**

Two degrees of freedom (one translation and one rotation) are considered at each end of the member. Hence, there are four possible degrees of freedom for this member and hence the resulting stiffness matrix is of the order $4 \cdot 4$ . In this method counterclockwise moments and counterclockwise rotations are taken as positive. The positive sense of the translation and rotation are also shown in the figure. Displacements are considered as positive in the direction of the co-ordinate axis. The elements of the stiffness matrix indicate the forces exerted on the member by the restraints at the ends of the member when unit displacements are imposed at each end of the member. Let us calculate the forces developed in the above beam member when unit displacement is imposed along each degree of freedom holding all other displacements to zero. Now impose a unit displacement along $y'$ axis at $j$ end of the member while holding all other displacements to zero as shown in Fig. 27.3a. This displacement causes both shear and moment in the beam. The restraint actions are also shown in the figure. By definition they are elements of the member stiffness matrix. In particular they form the first column of element stiffnessmatrix.

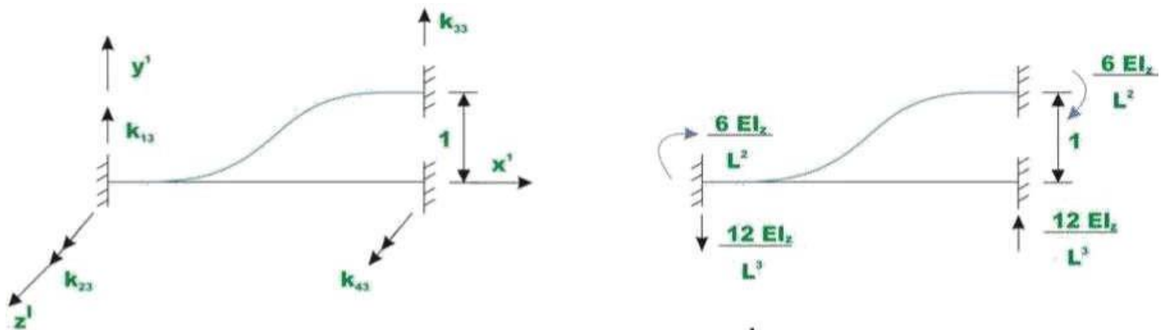
In Fig. 27.3b, the unit rotation in the positive sense is imposed at $j$ end of th beam while holding all other displacements to zero. The restraint actions are shown in the figure. The restraint actions at ends are calculated referring to tables given in lesson …



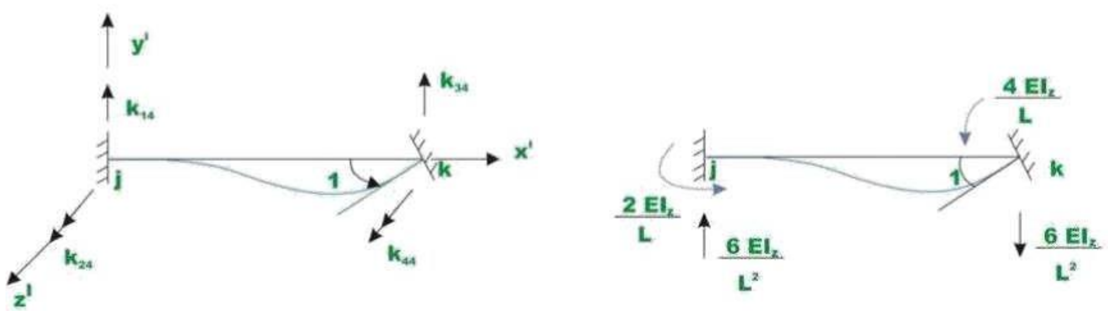

( a ) Unit translation along y' at end j




( b ) Unit rotation about z' at end j




**( c ) Unit displacement along y'at end k**




**( d ) Unit rotation about z'at end k**

**Fig. 27.3 Computation of beam stiffness matrix**

In Fig. 27.3c, unitdisplacementalong $y'$ axisatend $k$ is imposed and corresponding restraint actions are calculated. Similarly in Fig. 27.3d, unit rotation about $z'$ axis at end $k$ is imposed and corresponding stiffness coefficients are calculated. Hence the member stiffness matrix for the beam memberis

$$
[k] = \begin{array}{cc}
\begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\
\left[\begin{array}{cc|cc}
\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\
\hline
-\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\
\frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L}
\end{array}\right] & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}
\end{array} \qquad (27.1)
$$

The stiffness matrix is symmetrical. The stiffness matrix is partitioned to separate the actions associated with two ends of the member. For continuous beam problem, if the supports are unyielding, then only rotational degree of freedom shown in Fig. 27.4 is possible. In such a case the first and the third rows and columns will be deleted. The reduced stiffness matrix will be,

$$[k] = \left[\begin{array}{c|c} \frac{4EI_z}{L} & \frac{2EI_z}{L} \\ \hline \frac{2EI_z}{L} & \frac{4EI_z}{L} \end{array}\right] \qquad (27.2)$$

Instead of imposing unit displacement along $y'$ at $j$ end of the member in Fig. 27.3a, apply a displacement $u'_1$ along $y'$ at $j$ end of the member as shown in Fig. 27.5a, holding all other displacements to zero. Let the restraining forces developed be denoted by $q_{11}, q_{21}, q_{31}$ and $q_{41}$.

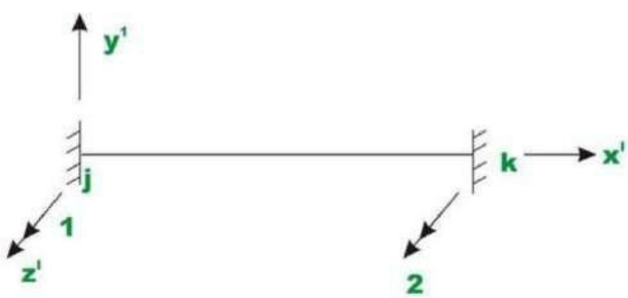
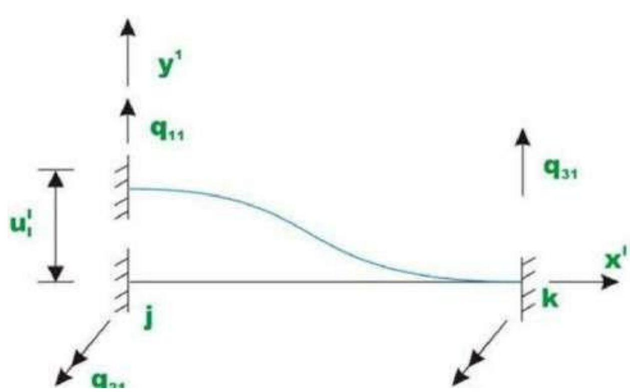



**Fig. 27.4**



Now, give displacements $u'_1, u'_2, u'_3$ and $u'_4$ simultaneously along displacement degrees of freedom 1,2,3 and 4 respectively. Let the restraining forces developed at member ends be $q_1, q_2, q_3$ and $q_4$ respectively as shown in Fig. 27.5b along respective degrees of freedom. Then by the principle of superposition, the force displacement relationship can be written as,

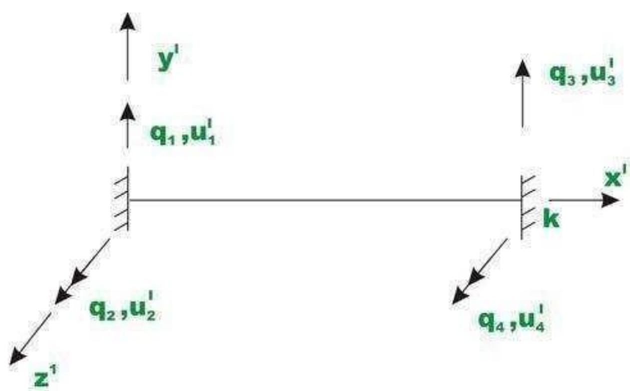



**Fig. 27.5 (b) Force - displacement relation**

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{bmatrix} \quad (27.4)$$

$$\{q\} = [k]\{u'\} \quad (27.5)$$

## BEAM (GLOBAL) STIFFNESS MATRIX.

The formation of structure (beam) stiffness matrix from its member stiffness matrices is explained with help of two span continuous beam shown in Fig. 27.6a. Note that no loading is shown on the beam. The orthogonal co-ordinate system $xyz$ denotes the global co-ordinatesystem.

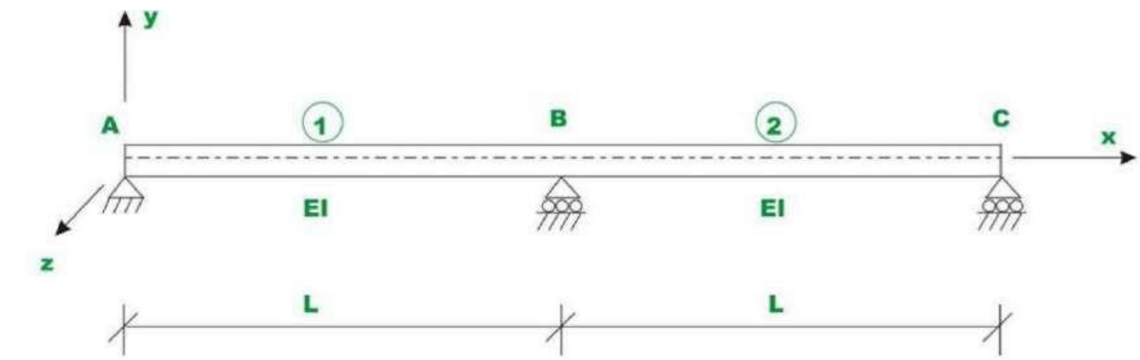



**Fig. 27.6a Continuous beam**

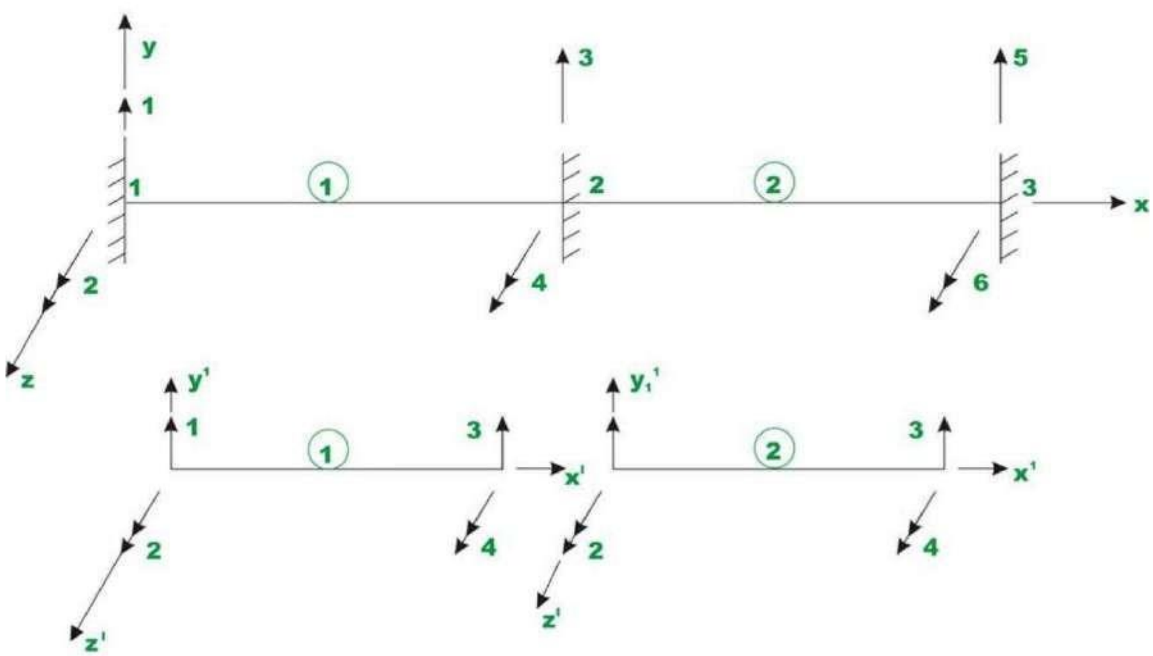



**Fig. 27.6 b**

For the case of continuous beam, the $x$ - and $x'$ - axes are collinear and other axes ( $y$ and $y'$ , $z$ and $z'$ ) are parallel to each other. Hence it is not requiredto

transform member stiffness matrix from local co-ordinate system to global coordinate system as done in the case of trusses. For obtaining the global stiffness matrix, first assume that all joints are restrained. The node and member numbering for the possible degrees of freedom are shown in Fig 27.6b. The continuous beam is divided into two beam members. For this member there are six possible degrees of freedom. Also in the figure, each beam member with its displacement degrees of freedom (in local co ordinate system) is also shown.

Since the continuous beam has the same moment of inertia and span, the member stiffness matrix of element 1 and 2 are the same. They are,

$$\begin{array}{l} \text{Global } d.o.f \quad 1 \qquad 2 \qquad 3 \qquad 4 \\ \text{Local } d.o.f \quad\; 1 \qquad 2 \qquad 3 \qquad 4 \end{array}$$

$$[k'] = \begin{array}{c} \begin{bmatrix} k'_{11} & k'_{12} & k'_{13} & k'_{14} \\ k'_{21} & k'_{22} & k'_{23} & k'_{24} \\ k'_{31} & k'_{32} & k'_{33} & k'_{34} \\ k'_{41} & k'_{42} & k'_{43} & k'_{44} \end{bmatrix} \end{array} \begin{array}{cc} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{array} \qquad (27.6a)$$

$$\begin{array}{l} \text{Global } d.o.f \quad 3 \qquad 4 \qquad 5 \qquad 6 \\ \text{Local } d.o.f \quad\; 1 \qquad 2 \qquad 3 \qquad 4 \end{array}$$

$$[k^2] = \begin{bmatrix} k^2_{11} & k^2_{12} & k^2_{13} & k^2_{14} \\ k^2_{21} & k^2_{22} & k^2_{23} & k^2_{24} \\ k^2_{31} & k^2_{32} & k^2_{33} & k^2_{34} \\ k^2_{41} & k^2_{42} & k^2_{43} & k^2_{44} \end{bmatrix} \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & 5 \\ 4 & 6 \end{array} \qquad (27.6b)$$

The local and the global degrees of freedom are also indicated on the top and side of the element stiffness matrix. This will help us to place the elements of the element stiffness matrix at the appropriate locations of the global stiffness matrix. The continuous beam has six degrees of freedom and hence the stiffness matrix is of the order $6 \times 6$ Let $[K]$ denotes the continuous beam stiffness matrix of order $6 \times 6$ From Fig. 27.6b, $[K]$ may be written as, *Member AB* (1)

$$[K]=\left[\begin{array}{cc|cc|cc}
k_{11}^{1} & k_{12}^{1} & k_{13}^{1} & k_{14}^{1} & 0 & 0 \\
k_{21}^{1} & k_{22}^{1} & k_{23}^{1} & k_{24}^{1} & 0 & 0 \\
\hline
k_{31}^{1} & k_{32}^{1} & k_{33}^{1}+k_{11}^{2} & k_{34}^{1}+k_{12}^{2} & k_{13}^{2} & k_{14}^{2} \\
k_{41}^{1} & k_{42}^{1} & k_{43}^{1}+k_{21}^{2} & k_{44}^{1}+k_{22}^{2} & k_{23}^{2} & k_{24}^{2} \\
\hline
0 & 0 & k_{31}^{2} & k_{32}^{2} & k_{33}^{2} & k_{34}^{2} \\
0 & 0 & k_{41}^{2} & k_{42}^{2} & k_{43}^{2} & k_{44}^{2}
\end{array}\right] \tag{27.7}$$

*Member BC* (2)

The $4$ x $4$ upper left hand section receives contribution from member 1 $(AB)$ and

$4$ x $4$ lower right hand section of global stiffness matrix receives contribution frommember 2. The element of the global stiffness matrix corresponding to global degrees of freedom 3 and 4 [overlapping portion of equation $(27.7)$] receives element from both members 1 and 2.

## Formation of load vector.

Consider a continuous beam $ABC$ as shown in Fig. 27.7.

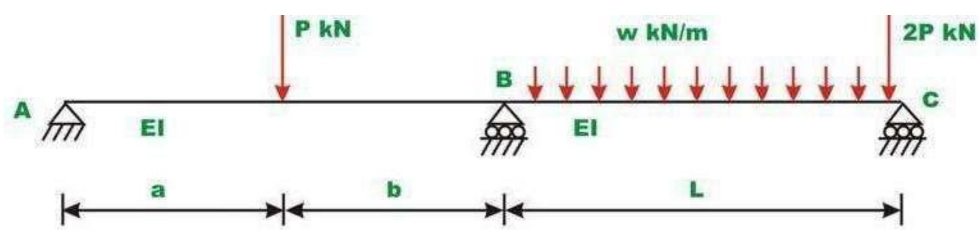
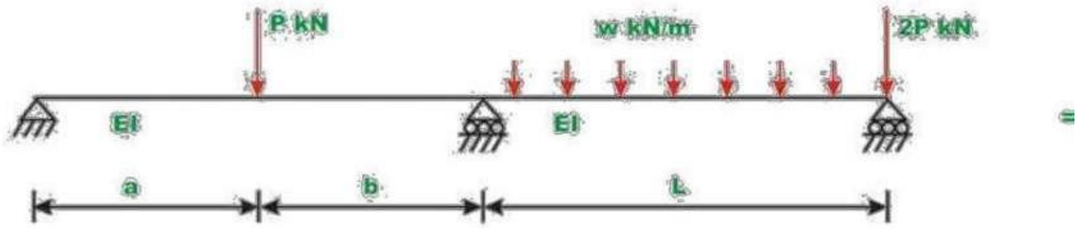
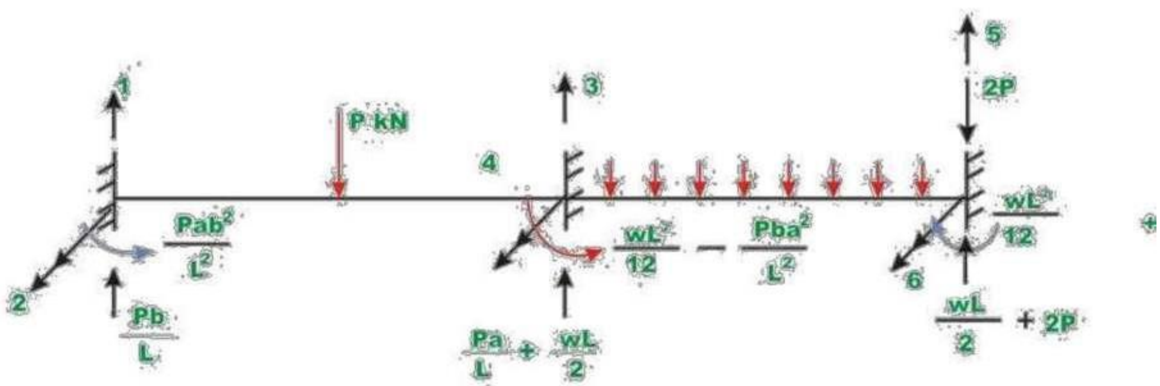



**Fig.27.7**

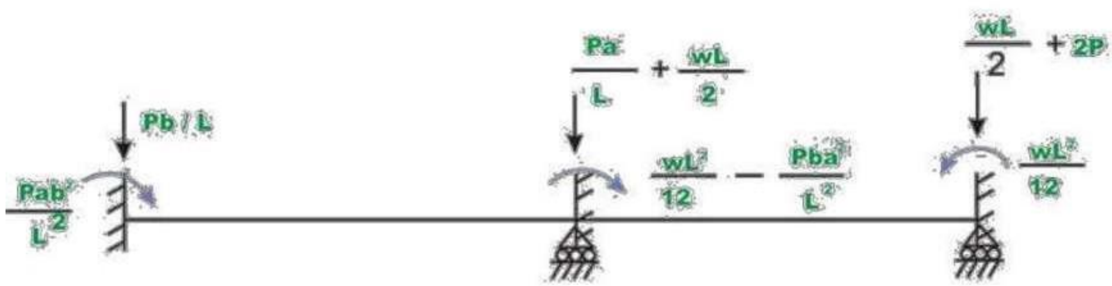
We have two types of load: member loads and joint loads. Joint loads could be handled very easily as done in case of trusses. Note that stiffness matrix of each member was developed for end loading only. Thus it is required to replace the member loads by equivalent joint loads. The equivalent joint loads must be evaluated such that the displacements produced by them in the beam should be the same as the displacements produced by the actual loading on the beam. This is evaluated by invoking the method of superposition.




**( a ) Actual beam with loading**




**( b ) Reaction in the restrained beam**




**( c ) Equivalent joint loads**

**Fig. 27.8**

The loading on the beam shown in Fig. 27.8(a), is equal to the sum of Fig.

27.8(b) and Fig. 27.8(c). In Fig. 27.8(c), the joints are restrained against displacements and fixed end forces are calculated. In Fig. 27.8(c) these fixed end actions are shown in reverse direction on the actual beam without any load.

Since the beam in Fig. 27.8(b) is restrained (fixed) against any displacement, the displacements produced by the joint loads in Fig. 27.8(c) must be equal to the displacement produced by the actual beam in Fig. 27.8(a). Thus the loads shown in Fig. 27.8(c) are the equivalent joint loads .Let, $p_1, p_2, p_3, p_4, p_5$ and $p_6$ be the

equivalent joint loads acting on the continuous beam along displacement degrees of freedom 1,2,3,4,5 and 6 respectively as shown in Fig. 27.8(b). Thus th global load vector is,

$$\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} -\dfrac{Pb}{L} \\ -\dfrac{Pab^2}{L^2} \\ -\left(\dfrac{Pa}{L}+\dfrac{wL}{2}\right) \\ -\left(\dfrac{wL^2}{12}-\dfrac{Pba^2}{L^2}\right) \\ -\left(\dfrac{wL}{2}+2P\right) \\ \dfrac{wL^2}{12} \end{Bmatrix} \qquad (27.8)$$

## Solution of equilibrium equations

After establishing the global stiffness matrix and load vector of the beam, the load displacement relationship for the beam can be writtenas,

$$\{P\} = [K]\{u\} \qquad (27.9)$$

where $\{P\}$ is the global load vector, $\{u\}$ is displacement vector and $[K]$ is the

global stiffness matrix. This equation is solved exactly in the similar manner as discussed in the lesson 24. In the above equation some joint displacements are known from support conditions. The above equation may be written as

$$\begin{Bmatrix} \{p_k\} \\ \{p_u\} \end{Bmatrix} = \begin{bmatrix} [k_{11}] & [k_{12}] \\ [k_{21}] & [k_{22}] \end{bmatrix} \begin{Bmatrix} \{u_u\} \\ \{u_k\} \end{Bmatrix} \qquad (27.10)$$

displacements. And $\{p_u\}$, $\{u_u\}$ denote respectively vector of unknown forces and unknown displacements respectively. Now expanding equation(27.10),

$$\{p_k\} = [k_{11}]\{u_u\} + [k_{12}]\{u_k\} \qquad (27.11a)$$

$$\{p_u\} = [k_{21}]\{u_u\} + [k_{22}]\{u_k\} \qquad (27.11b)$$

Since $\{u_k\}$ is known, from equation 27.11(a), the unknown joint displacements can be evaluated. And support reactions are evaluated from equation (27.11b), after evaluating unknown displacement vector.

Let $R_1, R_3$ and $R_5$ be the reactions along the constrained degrees of freedom as shown in Fig. 27.9a. Since equivalent joint loads are directly applied at the supports, they also need to be considered while calculating the actual reactions.

Thus,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = -\begin{Bmatrix} p_1 \\ p_3 \\ p_5 \end{Bmatrix} + [K_{21}]\{u_u\} \qquad (27.12)$$

The reactions may be calculated as follows. The reactions of the beam shown in Fig. 27.9a are equal to the sum of reactions shown in Fig. 27.9b, Fig. 27.9c and Fig. 27.9d.

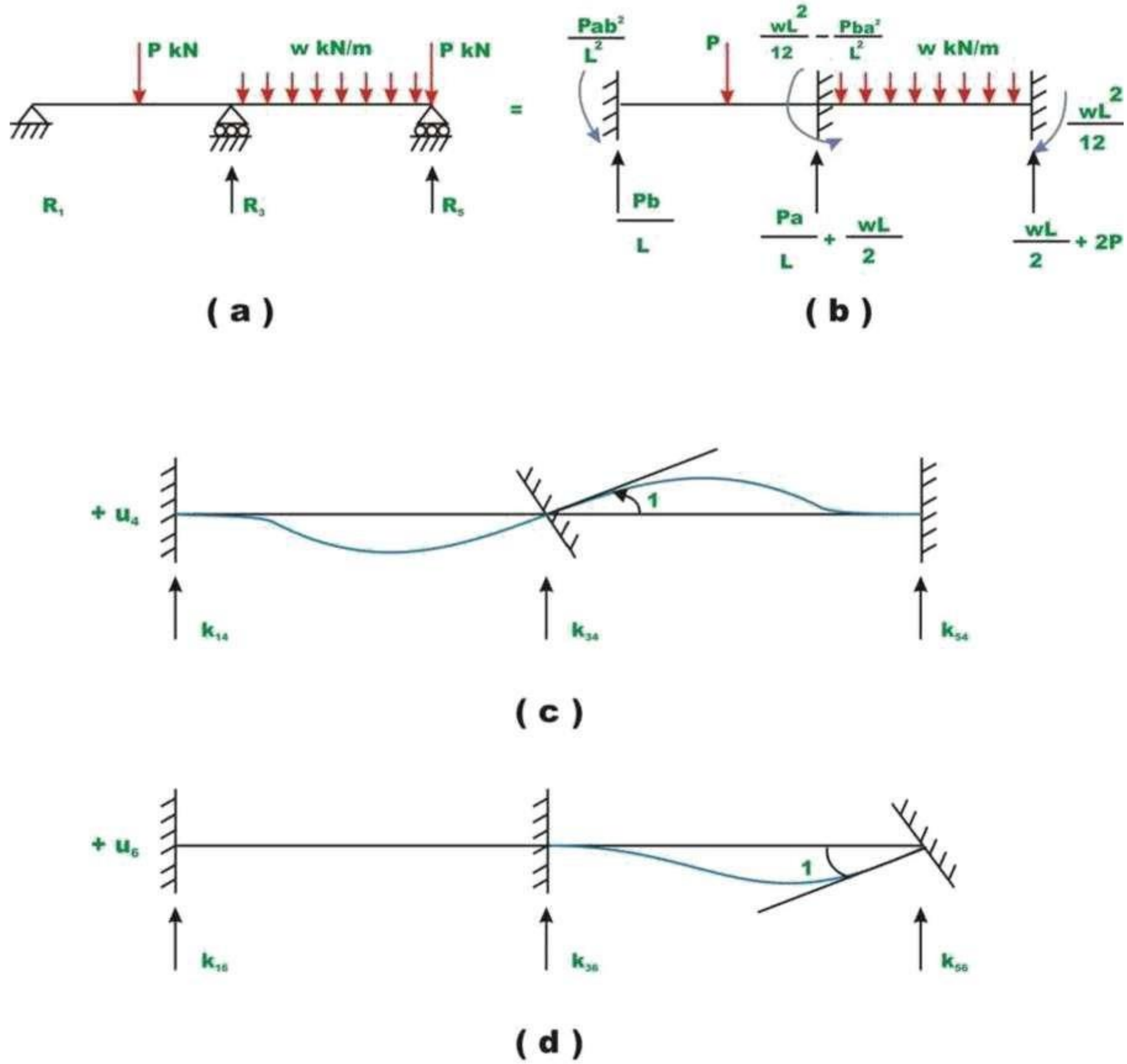

P kN
w kN/m
P kN
R₁
R₃
R₅
=
Pab²/L²
P
wL²/12 – Pba²/L²
w kN/m
wL²/12
Pb/L
Pa/L + wL/2
wL/2 + 2P
( a )
( b )
+ u₄
1
k₁₄
k₃₄
k₅₄
( c )
+ u₆
1
k₁₆
k₃₆
k₅₆
( d )

Fig. 27.9

From the method of superposition,

$$R_1 = \frac{Pb}{L} + K_{14}u_4 + K_{16}u_6 \tag{27.13a}$$

$$R_3 = \frac{Pa}{L} + K_{34}u_4 + K_{36}u_6 \tag{27.13b}$$

$$R_5 = \frac{wL}{2} + 2P + K_{54}u_4 + K_{56}u_6 \tag{27.13c}$$

or

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = \begin{Bmatrix} Pb/L \\ Pa/L \\ \frac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \tag{27.14a}$$

Equation (27.14a) may be written as,

$$\begin{Bmatrix} R_1 \\ R_3 \\ R_5 \end{Bmatrix} = -\begin{Bmatrix} Pb/L \\ Pa/L \\ \frac{wl}{2} + 2P \end{Bmatrix} + \begin{bmatrix} K_{14} & K_{16} \\ K_{34} & K_{36} \\ K_{54} & K_{56} \end{bmatrix} \begin{Bmatrix} u_4 \\ u_6 \end{Bmatrix} \tag{27.14b}$$

Member end actions $q_1, q_2, q_3, q_4$ are calculated as follows. For example consider the first element 1.

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} \frac{Pb}{L} \\ \frac{Pab^2}{L^2} \\ \frac{Pa}{L} \\ -\frac{Pa^2b}{L^2} \end{Bmatrix} + [K]_{element1} \begin{Bmatrix} 0 \\ u_2 \\ 0 \\ u_4 \end{Bmatrix} \tag{27.16}$$

In the next lesson few problems are solved to illustrate the method so far discussed.

In the last lesson, the procedure to analyse beams by direct stiffness method has been discussed. No numerical problems are given in that lesson. In this lesson, few continuous beam problems are solved numerically by direct stiffness method.

**Example 1**

Analyse the continuous beam shown in Fig. 28.1a. Assume that the supports are unyielding. Also assume that $EI$ is constant for all members.

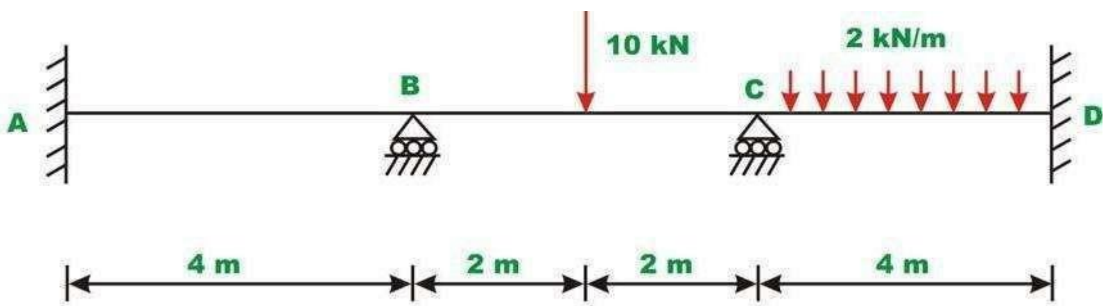



**Fig. 28.1a**

The numbering of joints and members are shown in Fig. 28.1b. The possible global degrees of freedom are shown in the figure. Numbers are put for the unconstrained degrees of freedom first and then that for constrained displacements.

The given continuous beam is divided into three beam elements Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. In the given problem some displacements are zero, i.e., $u_3=u_4=u_5=u_6=u_7=u_8=0$ from support conditions.

In the case of beams, it is not required to transform member stiffness matrix from local co-ordinate system to global co-ordinate system, as the two co-ordinate system are parallel to each other.

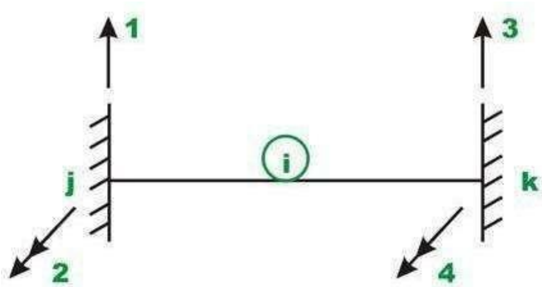



**Figure 28.1c**

First construct the member stiffness matrix for each member. This may be done from the fundamentals. However, one could use directly the equation (27.1) given in the previous lesson and reproduced below for the sakeconvenience.

$$[k]=\begin{array}{c} \begin{array}{cccc} \quad 1 \quad & \quad 2 \quad & \quad 3 \quad & \quad 4 \quad \end{array} \\ \left[\begin{array}{cc|cc} \frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} & -\frac{12EI_z}{L^3} & \frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{4EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{2EI_z}{L} \\ \hline -\frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} & \frac{12EI_z}{L^3} & -\frac{6EI_z}{L^2} \\ \frac{6EI_z}{L^2} & \frac{2EI_z}{L} & -\frac{6EI_z}{L^2} & \frac{4EI_z}{L} \end{array}\right] \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \qquad (27.1)$$

The degrees of freedom of a typical beam member are shown in Fig. 28.1c. Here equation (1) is used to generate element stiffness matrix.

**Member 1**: $L=4m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{array}{cccccc} Global\ d.o.f & 6 & 5 & 3 & 1 & \end{array}$$

$$[k^1]=EI_{zz}\left[\begin{array}{cccc} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{array}\right] \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \qquad (2)$$

Me

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitate assembling.

$$\begin{array}{cccccc} Global\ d.o.f & 3 & 1 & 4 & 2 & \end{array}$$

$$[k^2]=EI_{zz}\left[\begin{array}{cccc} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{array}\right] \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array} \qquad (3)$$

**Member 3**: $L=4m$, node points 3-4.

$$\begin{array}{c|cccc|c} \text{Global d.o.f} & 4 & 2 & 8 & 7 & \\ \hline [k^3] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} & & & & \begin{matrix} 4 \\ 2 \\ 8 \\ 7 \end{matrix} \end{array} \qquad (4)$$

The assembled global stiffness matrix of the continuous beam is of the order 8 x 8. The assembled global stiffness matrix may be written as,

$$[K] = EI_{zz} \begin{bmatrix} 2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\ 0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 & 0 & 0 \\ -0.375 & 0 & -0.1875 & 0.375 & 0 & 0 & 0.375 & -0.1875 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 & 0 & 0 \\ 0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\ 0 & -0.375 & 0 & -0.1875 & 0 & 0 & -0.375 & 0.1875 \end{bmatrix} \qquad (5)$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.1d. The displacement degrees of freedom are also shown in Fig. 28.1d.

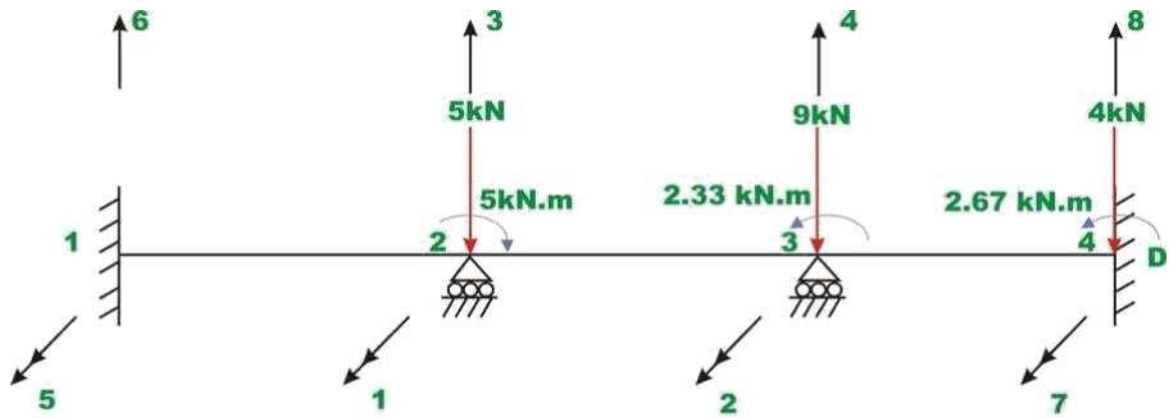



**Fig. 28.1 (d) Equivalent joint loads**

hus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} \qquad (6)$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} -5 \\ 2.33 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = EI_{zz} \left[ \begin{array}{cc|cccccc} 2.0 & 0.5 & 0.0 & -0.375 & 0.5 & 0.375 & 0 & 0 \\ 0.5 & 2.0 & 0.375 & 0 & 0 & 0 & 0.5 & -0.375 \\ \hline 0 & 0.375 & 0.375 & -0.187 & -0.375 & -0.187 & 0 & 0 \\ -0.375 & 0 & -0.187 & 0.375 & 0 & 0 & 0.375 & -0.187 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 & 0 & 0 \\ 0.375 & 0 & -0.187 & 0 & 0.375 & 0.187 & 0 & 0 \\ 0 & 0.5 & 0 & 0.375 & 0 & 0 & 1.0 & -0.375 \\ 0 & -0.375 & 0 & -0.187 & 0 & 0 & -0.375 & 0.187 \end{array} \right] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{Bmatrix} \tag{7}$$

where $\{p\}$ is the joint load vector, $\{u\}$ is displacement vector. $u_1$ and

We know that $u_3 = u_4 = u_5 = u_6 = u_7 = u_8 = 0$ . Thus solving for unknowns

$u_2$, yields

$$\begin{Bmatrix} -5 \\ 2.33 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{3.75 EI_{zz}} \begin{bmatrix} 2.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} -5 \\ 2.333 \end{Bmatrix} \tag{9}$$

$$= \frac{1}{EI_{zz}} \begin{Bmatrix} -2.977 \\ 1.909 \end{Bmatrix}$$

Thus displacements are,

$$u_1 = \frac{-2.977}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{1.909}{EI_{zz}} \tag{10}$$

The unknown joint loads are given by,

The actual reactions at the supports are calculated as,

$$\begin{Bmatrix} R_3 \\ R_4 \\ R_5 \\ R_6 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} p_3^F \\ p_4^F \\ p_5^F \\ p_6^F \\ p_7^F \\ p_8^F \end{Bmatrix} + \begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{Bmatrix} = \begin{Bmatrix} 5 \\ 9 \\ 0 \\ 0 \\ -2.67 \\ 4 \end{Bmatrix} + \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \\ 0.955 \\ -0.715 \end{Bmatrix} = \begin{Bmatrix} 5.716 \\ 10.116 \\ -1.489 \\ -1.116 \\ -1.715 \\ 3.284 \end{Bmatrix} \tag{12}$$

$$= \begin{Bmatrix} 0.715 \\ 1.116 \\ -1.488 \\ -1.116 \end{Bmatrix}$$

## Member end actions for element 1

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -2.977 \end{Bmatrix}$$

$$= \begin{Bmatrix} -1.116 \\ -1.488 \\ 1.116 \\ -2.977 \end{Bmatrix} \tag{13}$$

## Member end actions for element 2

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = +EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ -2.977 \\ 0 \\ 1.909 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.6 \\ 2.98 \\ 5.4 \\ -4.58 \end{Bmatrix} \qquad (14)$$

## Member end actions for element 3

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} 4.0 \\ 2.67 \\ 4.0 \\ -2.67 \end{Bmatrix} + EI_{zz} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} 0 \\ 1.909 \\ 0 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 4.72 \\ 4.58 \\ 3.28 \\ -1.72 \end{Bmatrix} \qquad (15)$$

## Example 2

Analyse the continuous beam shown in Fig. 28.2a. Assume that the supports are unyielding. Assume $EI$ to be constant for all members.

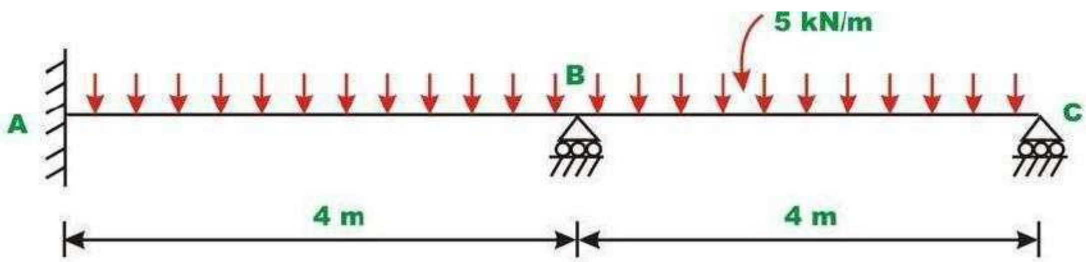



**Fig. 28.2a**

The numbering of joints and members are shown in Fig. 28.2b. The global degrees of freedom are also shown in the figure.

The given continuous beam is divided into two beam elements. Two degrees of freedom (one translation and one rotation) are considered at each end of the member. In the above figure, double headed arrows denote rotations and single headed arrow represents translations. Also it is observed that displacements $u_3=u_4=u_5=u_6=0$ from support conditions.

First construct the member stiffness matrix for each member.

**Member 1**: $L=4m$, node points 1-2.

The member stiffness matrix for all the members are the same, as the length and flexural rigidity of all members is the same.

$$\begin{array}{lc} \text{Global d.o.f} & \begin{array}{cccc} 6 & 5 & 3 & 1 \end{array} \\ [k'] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 6 \\ 5 \\ 3 \\ 1 \end{array} \end{array} \qquad (1)$$

On the member stiffness matrix, the corresponding global degrees of freedom are indicated to facilitateassembling.

**Member 2**: $L=4m$, node points 2-3.

$$
\begin{array}{cc}
 & \begin{array}{cccc} 3 & 1 & 4 & 2 \end{array} \\
[k^2] = EI_{zz} & \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1.0 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1.0 \end{bmatrix} \begin{array}{c} 3 \\ 1 \\ 4 \\ 2 \end{array}
\end{array}
\quad (2)
$$

Global d.o.f

The assembled global stiffness matrix of the continuous beam is of order $6 \cdot 6$.

The assembled global stiffness matrix may be written as,

$$
[K] = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \quad (3)
$$

Now it is required to replace the given members loads by equivalent joint loads. The equivalent loads for the present case is shown in Fig. 28.2c. The displacement degrees of freedom are also shown in figure.

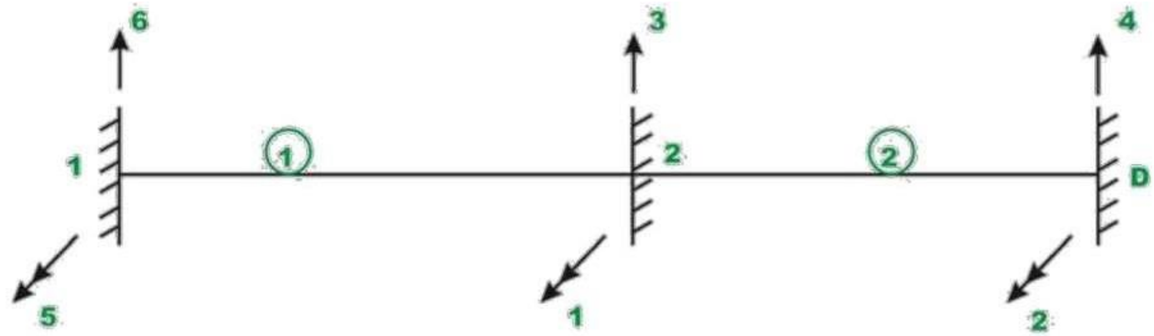



**Fig. 28.2b Node and member**

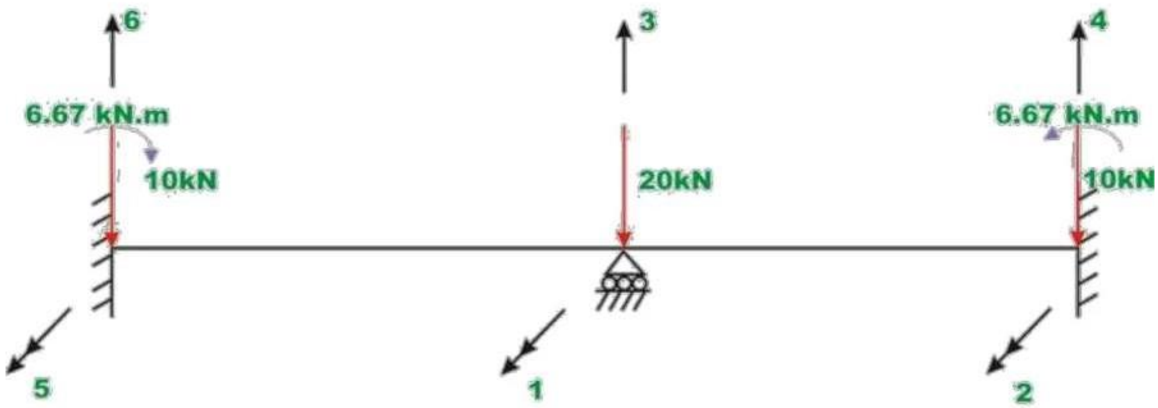



**Fig. 28.2c Equivalent joint loads**

Thus the global load vector corresponding to unconstrained degree of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} \qquad (4)$$

Writing the load displacement relation for the entire continuous beam,

$$\begin{Bmatrix} 0 \\ 6.67 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2 & 0.5 & 0 & -0.375 & 0.5 & 0.375 \\ 0.5 & 1.0 & 0.375 & -0.375 & 0 & 0 \\ 0 & 0.375 & 0.375 & -0.1875 & -0.375 & -0.1875 \\ -0.375 & -0.375 & -0.1875 & 0.1875 & 0 & 0 \\ 0.5 & 0 & -0.375 & 0 & 1.0 & 0.375 \\ 0.375 & 0 & -0.1875 & 0 & 0.375 & 0.1875 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix} \quad (5)$$

We know that $u_3=u_4=u_5=u_6=0$ Thus solving for unknowns $u_1$ and $u_2$ , yields

$$\begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 2.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \quad (6)$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{1.75 EI_{zz}} \begin{bmatrix} 1.0 & -0.5 \\ -0.5 & 2.0 \end{bmatrix} \begin{Bmatrix} 0 \\ 6.67 \end{Bmatrix}$$

$$= \frac{1}{EI_{zz}} \begin{Bmatrix} -1.905 \\ 7.62 \end{Bmatrix} \quad (7)$$

Thus displacements are,

$$u_1 = \frac{-1.905}{EI_{zz}} \quad \text{and} \quad u_2 = \frac{7.62}{EI_{zz}}$$

The unknown joint loads are given by,

$$\begin{Bmatrix} p_3 \\ p_4 \\ p_5 \\ p_6 \end{Bmatrix} = EI_{zz} \begin{bmatrix} 0 & 0.375 \\ -0.375 & -0.375 \\ 0.5 & 0 \\ 0.375 & 0 \end{bmatrix} \frac{1}{EI_{zz}} \begin{Bmatrix} -1.905 \\ 7.62 \end{Bmatrix}$$

## Example 30.1

Analyze the rigid frame shown in Fig 30.4a by direct stiffness matrix method.

Assume $E=200 GPa; I_{ZZ}=1.33 \cdot 10^{-4} m^4$ and $A=0.04 m^2$. The flexural rigidity $EI$ and axial rigidity $EA$ are the same for both thebeams.

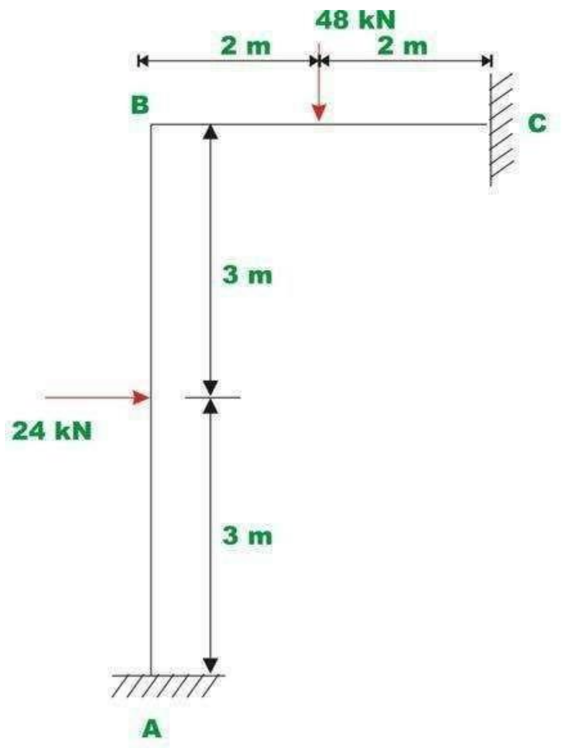



**Fig. 30.4a Rigid Frame.**

## Solution:

The plane frame is divided in to two beam elements as shown in Fig. 30.4b. The numbering of joints and members are also shown in Fig. 30.3b. Each node has three degrees of freedom. Degrees of freedom at all nodes are also shown in the figure. Also the local degrees of freedom of beam element are shown in the figure asinset.

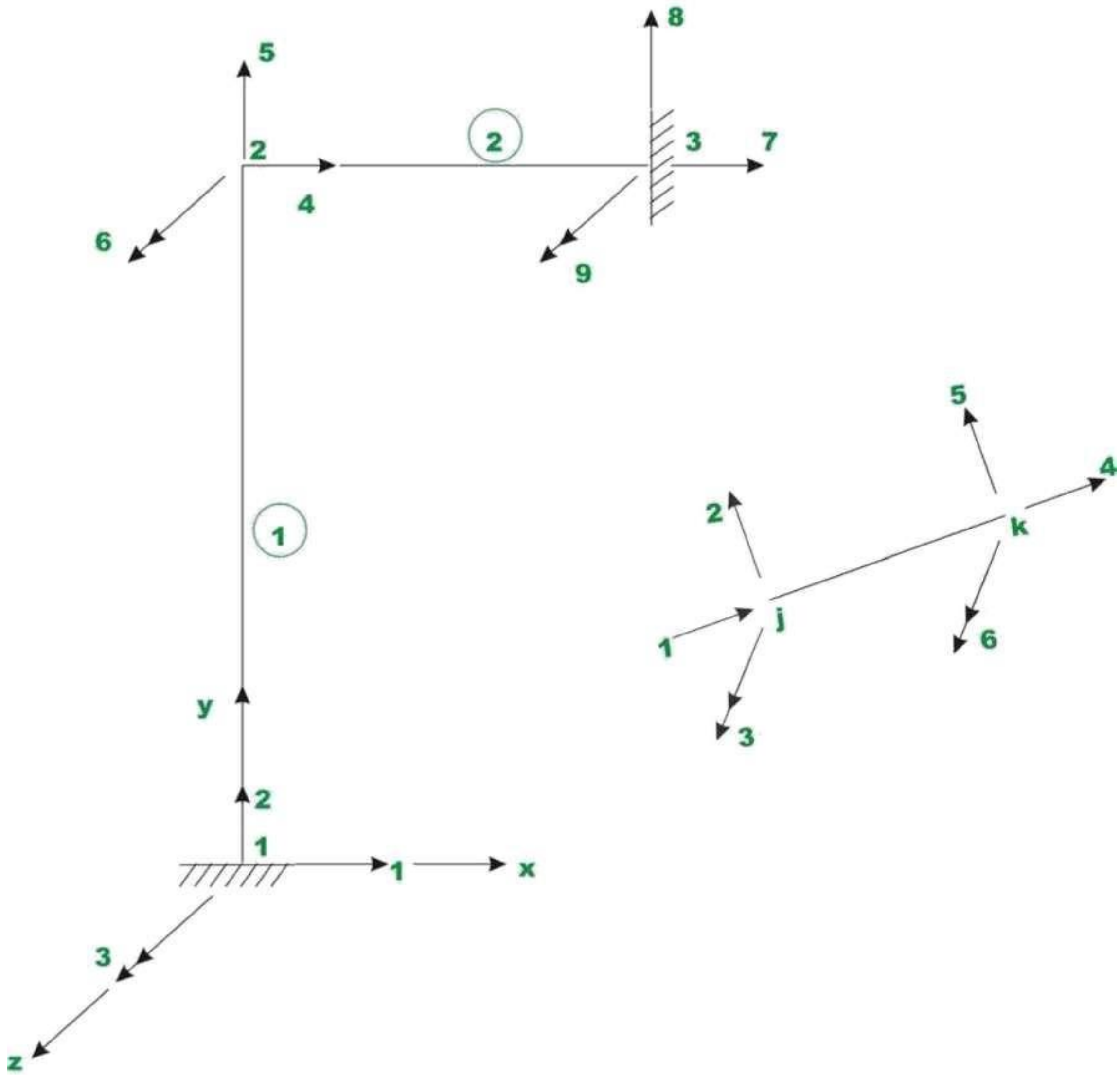



**Fig. 30.4b Node and member numbering.**

Formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. The origin of the global co-ordinate system is taken at node 1. Here the element stiffness matrix in global co-ordinates is only given.

**Member 1:** $L=6m$; $\theta=90°$ node points 1-2; $l=0$ and $m=1$.

$$[k^1]=[T]^T[k'][T]$$

$$[k^1]=\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\ 0 & 1.333\times10^6 & 0 & 0 & -1.333\times10^6 & 0 \\ 4.44\times10^3 & 0 & 17.78\times10^3 & 4.44\times10^3 & 0 & 8.88\times10^3 \\ 1.48\times10^3 & 0 & 4.44\times10^3 & 1.48\times10^3 & 0 & 4.44\times10^3 \\ 0 & -1.333\times10^6 & 0 & 0 & 1.333\times10^6 & 0 \\ 4.44\times10^3 & 0 & 8.88\times10^3 & 4.44\times10^3 & 0 & 17.78\times10^3 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

(1)

**Member 2:** $L=4\ m$ ; $\theta=0°$ ; node points 2-3 ; $l=1$ and $m=0$.

$$[k^2]=[T]^T[k'][T]$$

$$=\begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 2.0\times10^6 & 0 & 0 & -2.0\times10^6 & 0 & 0 \\ 0 & 5\times10^3 & 10\times10^3 & 0 & -5\times10^3 & 10\times10^3 \\ 0 & 10\times10^3 & 26.66\times10^3 & 0 & -10\times10^3 & 8.88\times10^3 \\ -2.0\times10^6 & 0 & 0 & 2.0\times10^6 & 0 & 0 \\ 0 & -5\times10^3 & -10\times10^3 & 0 & 5\times10^3 & -10\times10^3 \\ 0 & 10\times10^3 & 8.88\times10^3 & 0 & -10\times10^3 & 26.66\times10^3 \end{bmatrix} \end{array} \begin{array}{c} \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

(2)

The assembled global stiffness matrix $[K]$ is of the order $9\times9$. Carrying out assembly in the usual manner, we get,

$$[K]=\begin{bmatrix} 1.48 & 0 & -4.44 & -1.48 & 0 & -4.44 & 0 & 0 & 0 \\ 0 & 1333.3 & 0 & 0 & -1333.3 & 0 & 0 & 0 & 0 \\ -4.44 & 0 & 17.77 & 4.44 & 0 & 8.88 & 0 & 0 & 0 \\ -1.48 & 0 & 4.44 & 2001.5 & 0 & 4.44 & -2000 & 0 & 0 \\ 0 & -1333.3 & 0 & 0 & 1338.3 & 10 & 0 & -5 & 10 \\ -4.44 & 0 & 8.88 & 4.44 & 10 & 44.44 & 0 & -10 & 13.33 \\ 0 & 0 & 0 & -2000 & 0 & 0 & 2000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & -10 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 & 10 & 13.33 & 0 & -10 & 26.66 \end{bmatrix} \quad (3)$$

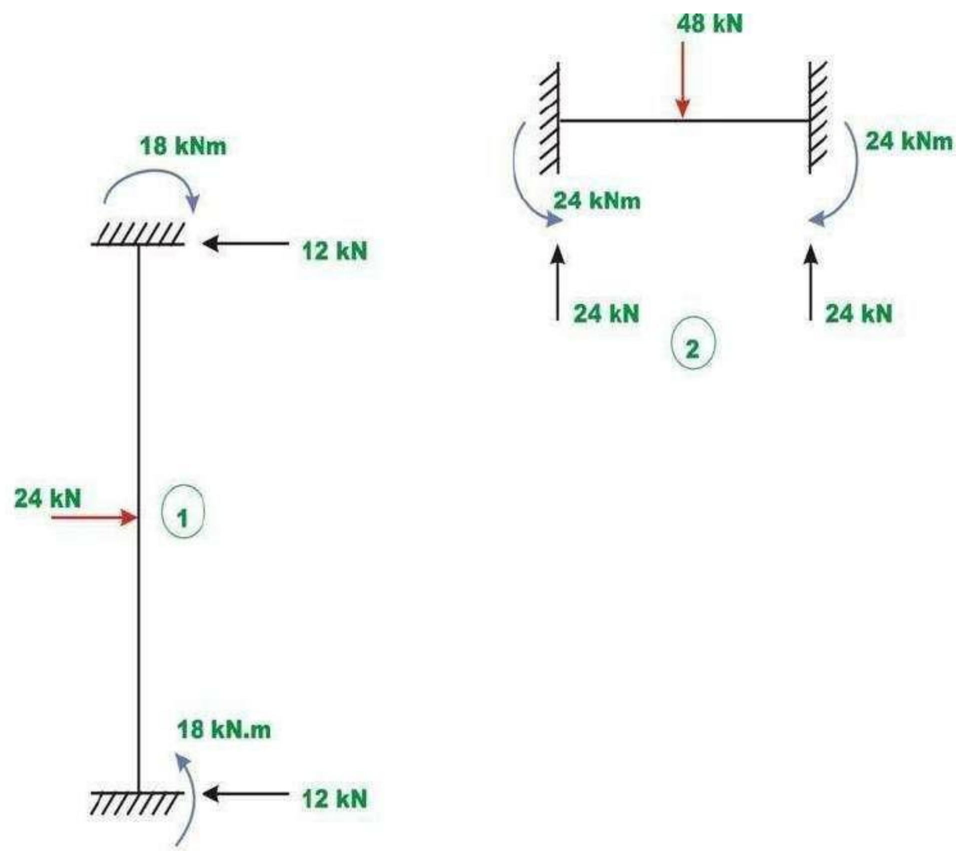



**Fig. 30.4c Fixed end action due to external load in element (1) and (2)**

The load vector corresponding to unconstrained degrees of freedom is (vide 30.4d),

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \end{Bmatrix} = \begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} \tag{4}$$

In the given frameconstraint degrees of freedom are $u_1, u_2, u_3, u_7, u_8, u_9$.

Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load-displacement relationship for only unconstrained degree of freedom,

$$\begin{Bmatrix} 12 \\ -24 \\ -6 \end{Bmatrix} = 10^3 \begin{bmatrix} 2001.5 & 0 & 4.44 \\ 0 & 1338.3 & 10 \\ 4.44 & 10 & 44.44 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} \tag{5}$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \end{Bmatrix} = \begin{Bmatrix} 6.28 \times 10^{-6} \\ -1.695 \times 10^{-5} \\ -0.13 \times 10^{-3} \end{Bmatrix} \tag{6}$$

$u_4 = 6.28 \times 10^{-6}\,\text{m.}, \quad u_5 = -1.695 \times 10^{-5}$

Let $R_1, R_2, R_3, R_7, R_8, R_9$ be the support reactions along degrees of freedom $1, 2, 3, 7, 8, 9$ respectively (vide Fig. 30.4e). Support reactions are calculated by

**Example 30.2**

Analyse the rigid frame shown in Fig 30.5a by direct stiffness matrix method.

Assume $E=200$ GPa ; $I_{ZZ}=1.33\cdot10^{-5}$ m$^4$ and $A=0.01 m^2$. The flexural rigidity $EI$ and axial rigidity $EA$ are the same for all beams.

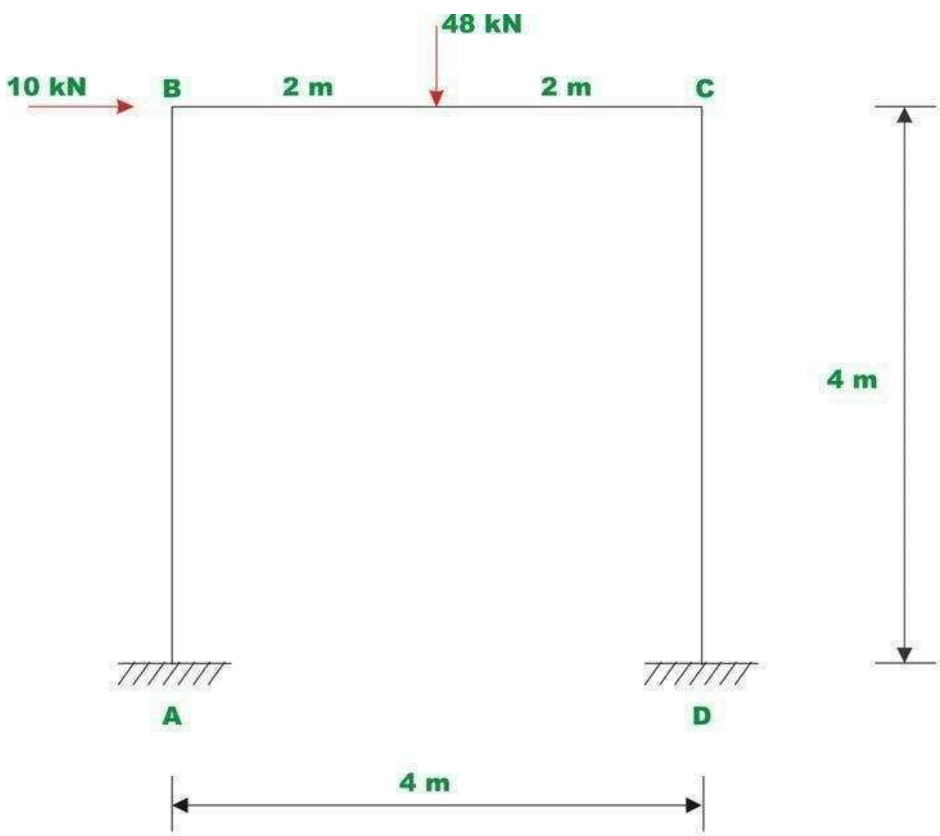



**Fig. 30.5a Rigid Frame of Example 30.2**

**Solution:**

The plane frame is divided in to three beam elements as shown in Fig. 30.5b. The numbering of joints and members are also shown in Fig. 30.5b. The possible degrees of freedom at nodes are also shown in the figure. The origin of the global co- ordinate system is taken at $A$(node1).

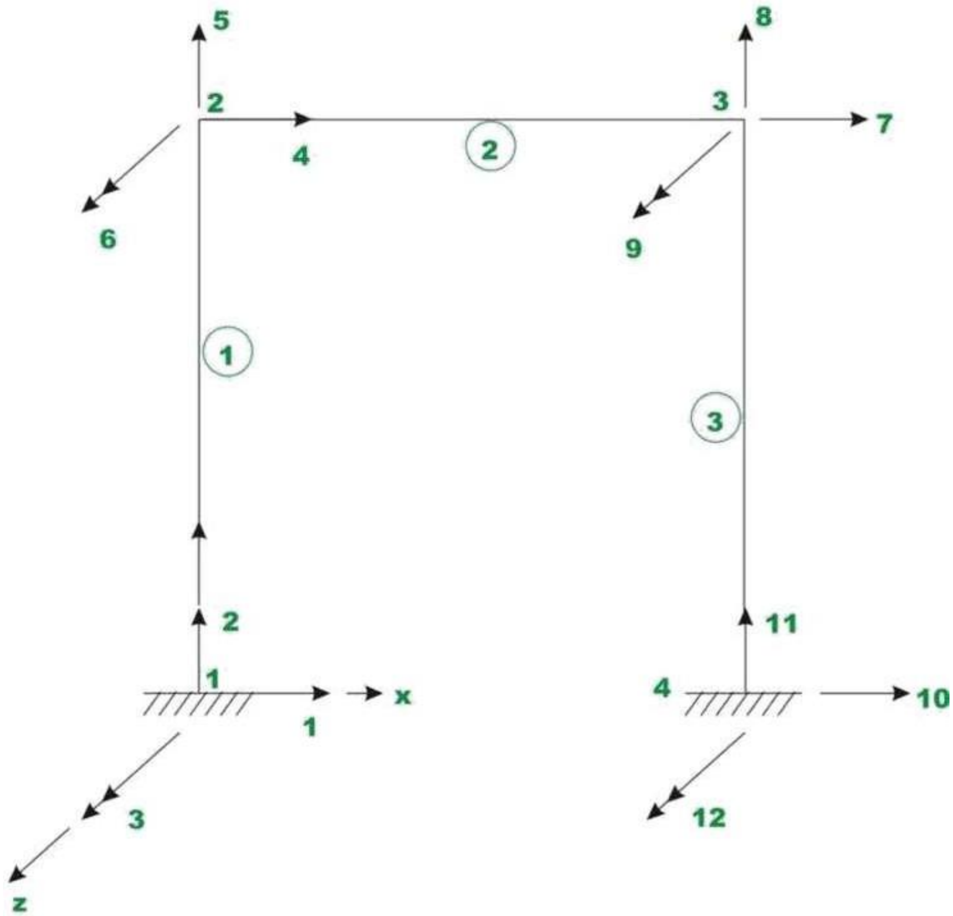



**Fig. 30.5b Node and Member numbering.**

Now formulate the element stiffness matrix in local co-ordinate system and then transform it to global co-ordinate system. In the present case three degrees of freedom are considered at each node.

**Member 1:** $L = 4\ m$ ; $\theta = 90°$ ; node points 1-2 ; $l = \frac{x_2 - x_1}{L} = 0$ and $m = \frac{y_2 - y_1}{L} = 1$.

The following terms are common for all elements.

$$\frac{AE}{L} = 5\times10^5 \text{ kN/m}; \quad \frac{6EI}{L^2} = 9.998\times10^2 \text{kN}$$

$$\frac{12EI}{L^3} = 4.999\times10^2 \text{ kN/m}; \quad \frac{4EI}{L} = 2.666\times10^3 \text{kN.m}$$

$$\left[k^1\right] = [T]^T [k'][T]$$

$$= \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.50\times10^3 & 0 & -1\times10^3 & -0.50\times10^3 & 0 & -1\times10^3 \\ 0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\ -1\times10^3 & 0 & 2.66\times10^3 & 1\times10^3 & 0 & 1.33\times10^3 \\ -0.50\times10^3 & 0 & 1\times10^3 & 0.50\times10^3 & 0 & 1\times10^3 \\ 0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\ -1\times10^3 & 0 & 1.33\times10^3 & 1\times10^3 & 0 & 2.66\times10^3 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

(1)

**Member 2:** $L = 4\ m$ ; $\theta = 0°$ node points 2-3; $l = 1$ and $m = 0$.

$$\left[k^2\right] = [T]^T [k'][T]$$

$$= \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} 5.0\times10^5 & 0 & 0 & -5.0\times10^6 & 0 & 0 \\ 0 & 0.5\times10^3 & 1\times10^3 & 0 & -0.5\times10^3 & 1\times10^3 \\ 0 & 1\times10^3 & 2.666\times10^3 & 0 & -1\times10^3 & 1.33\times10^3 \\ -5.0\times10^6 & 0 & 0 & 5.0\times10^6 & 0 & 0 \\ 0 & -0.5\times10^3 & -1\times10^3 & 0 & 0.5\times10^3 & -1\times10^3 \\ 0 & 1\times10^3 & 1.33\times10^3 & 0 & -1\times10^3 & 2.666\times10^3 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

(2)

**Member 3:** $L = 4\ m$ ; $\theta = 270°$ ; node points 3-4 ; $l = \dfrac{x_2 - x_1}{L} = 0$ and $m = \dfrac{y_2 - y_1}{L} = -1$.

$$\left[k^3\right] = [T]^T [k'][T]$$

$$= \begin{array}{c} \begin{array}{cccccc} 7 & 8 & 9 & 10 & 11 & 12 \end{array} \\ \begin{bmatrix} 0.50\times10^3 & 0 & 1\times10^3 & -0.50\times10^3 & 0 & 1\times10^3 \\ 0 & 5\times10^5 & 0 & 0 & -5\times10^5 & 0 \\ 1\times10^3 & 0 & 2.66\times10^3 & -1\times10^3 & 0 & 1.33\times10^3 \\ -0.50\times10^3 & 0 & -1\times10^3 & 0.50\times10^3 & 0 & -1\times10^3 \\ 0 & -5\times10^5 & 0 & 0 & 5\times10^5 & 0 \\ 1\times10^3 & 0 & 1.33\times10^3 & -1\times10^3 & 0 & 2.66\times10^3 \end{bmatrix} \end{array} \begin{array}{c} 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{array}$$

(3)

The assembled global stiffness matrix $[K]$ is of the order $12 \times 12$. Carrying out assembly in the usual manner, we get,

$$[K] = 10^3 \times \left[\begin{array}{ccc|cccccc|ccc}
0.50 & 0 & -1.0 & -0.50 & 0 & -1.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 500 & 0 & 0 & -500 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1.0 & 0 & 2.66 & 1.0 & 0 & 1.33 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
-0.50 & 0 & 1.0 & 500.5 & 0 & 1.0 & -500 & 0 & 0 & 0 & 0 & 0 \\
0 & -500 & 0 & 0 & 500.5 & 1.0 & 0 & -0.50 & 1.0 & 0 & 0 & 0 \\
-1.0 & 0 & 1.33 & 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 & 0 & 0 & 0 \\
0 & 0 & 0 & -500 & 0 & 0 & 500.5 & 0 & 1.0 & -0.5 & 0 & 1.0 \\
0 & 0 & 0 & 0 & -0.5 & -1.0 & 0 & 500.5 & -1.0 & 0 & -500 & 0 \\
0 & 0 & 0 & 0 & 1.0 & 1.33 & 1.0 & -1.0 & 5.33 & -1.0 & 0 & 1.33 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & -0.5 & 0 & -1.0 & 0.5 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -500 & 0 & 0 & 500 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1.0 & 0 & 1.33 & -1 & 0 & 2.66
\end{array}\right] \quad (4)$$

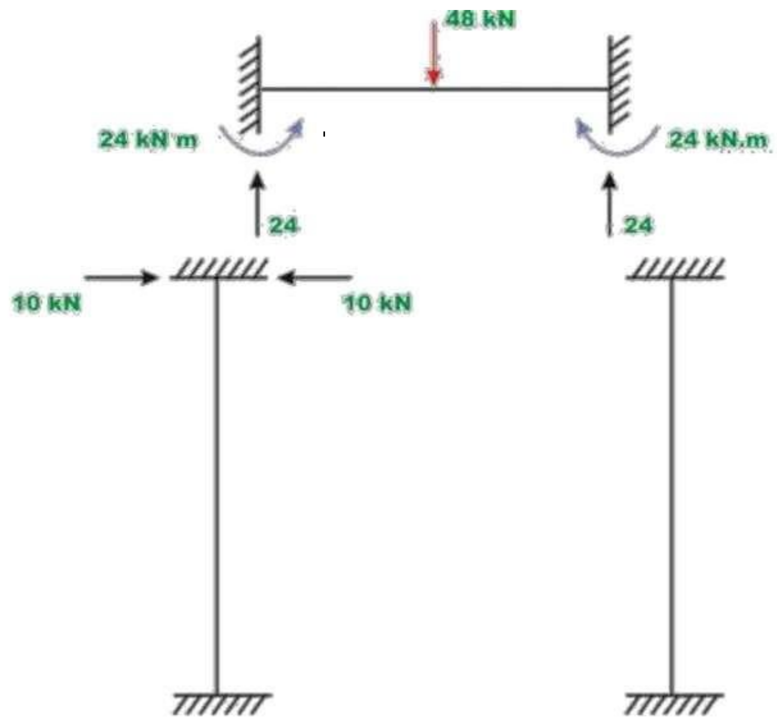



**Fig . 30.5c Fixed end action due to external load.**

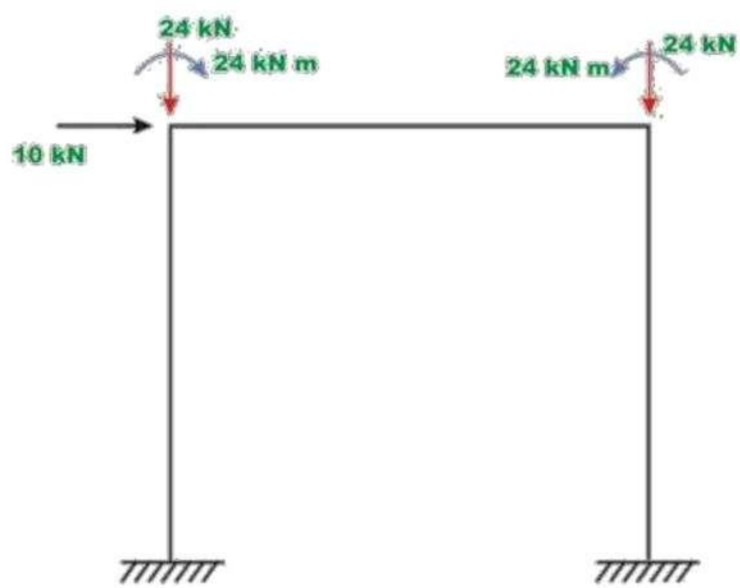



**Fig. 30.5d Equivalent joint loads.**

The load vector corresponding to unconstrained degrees of freedom is,

$$\{p_k\} = \begin{Bmatrix} p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \\ p_9 \end{Bmatrix} = \begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} \quad (5)$$

In the given frame, constraint (known) degrees of freedom are $u_1, u_2, u_3, u_{10}, u_{11}, u_{12}$. Eliminating rows and columns corresponding to constrained degrees of freedom from global stiffness matrix and writing load displacement relationship,

$$\begin{Bmatrix} 10 \\ -24 \\ -24 \\ 0 \\ -24 \\ 24 \end{Bmatrix} = 10^3 \begin{bmatrix} 500.5 & 0 & 1.0 & -500 & 0 & 0 \\ 0 & 500.5 & 1.0 & 0 & -0.5 & 1.0 \\ 1.0 & 1.0 & 5.33 & 0 & -1.0 & 1.33 \\ -500 & 0 & 0 & 500.5 & 0 & 1 \\ 0 & -0.5 & -1 & 0 & 500.5 & -1 \\ 0 & 1 & 1.33 & 1 & -1 & 5.33 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} \quad (6)$$

Solving we get,

$$\begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix} = \begin{Bmatrix} 1.43\times10^{-2} \\ -3.84\times10^{-5} \\ -8.14\times10^{-3} \\ 1.43\times10^{-2} \\ -5.65\times10^{-5} \\ 3.85\times10^{-3} \end{Bmatrix} \quad (7)$$

3,10,11,12 respectively. Support reactions are calculated by
Let $R_1, R_2, R_3, R_{10}, R_{11}, R_{12}$ be the support reactions along degrees of freedom 1, 2,

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} p_1{}^F \\ p_2{}^F \\ p_3{}^F \\ p_{10}{}^F \\ p_{11}{}^F \\ p_{12}{}^F \end{Bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \begin{bmatrix} -0.5 & 0 & -1.0 & 0 & 0 & 0 \\ 0 & -500 & 0 & 0 & 0 & 0 \\ 1.0 & 0 & 1.33 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & -1.0 \\ 0 & 0 & 0 & 0 & -500 & 0 \\ 0 & 0 & 0 & 1.0 & 0 & 1.33 \end{bmatrix} \end{array} \begin{Bmatrix} u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \end{Bmatrix}$$

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_{10} \\ R_{11} \\ R_{12} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} = \begin{Bmatrix} 0.99 \\ 19.71 \\ 3.43 \\ -10.99 \\ 28.28 \\ 19.42 \end{Bmatrix} \qquad (8)$$